Nonperturbative heat kernel and nonlocal effective action*

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Abstract

We present an overview of recent nonperturbative results in the theory of heat kernel and its late time asymptotics responsible for the infrared behavior of quantum effective action for massless theories. In particular, we derive the generalization of the Coleman-Weinberg potential to physical situations when the field is not homogeneous throughout the whole spacetime. This generalization represents a new nonlocal and nonperturbative action accounting for the effects of a transition domain between the spacetime interior and its infinity. In four dimensions these effects delocalize the logarithmic Coleman-Weinberg potential, while in $d > 4$ they are dominated by new powerlike and renormalization-independent nonlocal structure. Nonperturbative behavior of the heat kernel is also constructed in curved spacetime with asymptotically-flat geometry, and its conformal properties are analyzed for conformally invariant scalar field. The problem of disentangling the local cosmological term from nonlocal effective action is discussed.

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Contents

1. Introduction: heat kernel and effective action

It is widely recognized that nonlocal phenomena play a very important role in quantum physics. In contrast to low-energy vacuum polarization effects by massive quantum fields they characterize high-energy asymptotics in massive theories or infrared behavior in theories of massless fields. Even in models with well-established low-energy behavior like Einstein gravity these phenomena become increasingly interesting due to the attempts of resolving the cosmological constant and acceleration problems by means of nonlocal long-distance modifications of the theory. These modifications often call for nonperturbative treatment in view of the nonperturbative aspects of van Damm-Veltman-Zakharov discontinuity problem [1] and the presence of a hidden nonperturbative scale in gravitational models with extra dimensions [2].

On the other hand, nonlocalities also (and, moreover, primarily) arise in virtue of fundamental quantum effects of matter and graviton loops which, for instance, can play important role in gravitational radiation theory [3, 4] and cosmology [5]. Therefore, they can successfully compete with popular phenomenological mechanisms of infrared modifications, induced, say, by braneworld scenarios with extra dimensions [6, 7] or other models [8]. This makes nonperturbative analysis of nonlocal quantum effects very interesting and promising.

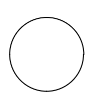
A basic tool for the description of these effects is the quantum effective action and the heat kernel technique of its calculation. For a generic theory of the field $\varphi(x)$ the classical action $S[\varphi]$ gives rise to the inverse propagator – the operator of linear field disturbances on the background of $\varphi(x)$

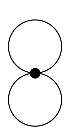
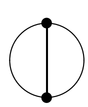
$$S[\varphi] \;\rightarrow\; F(\nabla)\delta(x,y) = \frac{\delta^2 S[\varphi]}{\delta\varphi(x)\delta\varphi(y)}. \tag{1.1}$$

Then the quantum effective action $\varGamma[\varphi]$ follows from its classical counterpart as a loop expansion in powers of $\hbar$

$$S[\varphi] \rightarrow \varGamma = S[\varphi] + \hbar\varGamma_{1-loop}[\varphi] + \hbar^2\varGamma_{2-loop}[\varphi] + ... \tag{1.2}$$

The first few orders of this expansion are graphically depicted below as Feynman graphs with the propagator – the Green's function of the operator (1.1) – and relevant vertices calculated on the background of a generic φ

$$\varGamma_{1-loop} = \frac{1}{2}\mathrm{Tr}\,\ln F(\nabla) = \frac{1}{2}$$  (1.3)

$$\varGamma_{2-loop} = \frac{1}{8}$$  $$+\frac{1}{12}$$  . (1.4)

The one-loop part (1.3) is peculiar in that it does not explicitly contain the vertices of the classical action (unless it is expanded in powers of the mean field φ) and given by the functional trace of the logarithm of $F(\nabla)$.

In local field theories without loss of generality the operator (1.1) has the form

$$F(\nabla) = \Box - V(x), \tag{1.5}$$

where $V(x)$ is some potential and

$$\Box = g^{\mu\nu}\nabla_\mu\nabla_\nu \tag{1.6}$$

is a covariant d'Alembertian in curved spacetime with metric $g_{\mu\nu}$.

A very efficient way of analyzing Feynman graphs of the form (1.3)-(1.4) on *generic field background*, i. e. with generic metric and potential, is based on the use of the heat kernel

$$K(s|\,x,y) = \exp(sF(\nabla))\,\delta(x,y). \tag{1.7}$$

It solves the heat equation with unit initial condition at $s = 0$

$$\frac{\partial K(s)}{\partial s} = F(\nabla)\,K(s), \quad K(0) = \mathbb{I} \tag{1.8}$$

and generates as a result of integration over the proper time parameter the main ingredient of the Feynman diagrammatic technique – the propagator of the operator (1.1)

$$G(x,y) \equiv \frac{1}{F(\nabla)}\,\delta(x,y) = -\int_0^\infty ds\,K(s|\,x,y) \tag{1.9}$$

and, in closed form, the one-loop effective action

$$\mathit{\Gamma}_{1-loop} = \frac{1}{2}\int_0^\infty \frac{ds}{s}\,\mathrm{Tr}\,K(s)\,, \tag{1.10}$$

$$\mathrm{Tr}\,K(s) = \int dx\,K(s|\,x,x). \tag{1.11}$$

The efficiency of the heat kernel and proper time method are based on the well-known and universal behavior of $K(s|x,y)$ for a generic second-order operator (1.5) at $s \to 0$. This limit is responsible for ultraviolet divergences and anomalies in field theory, renormalization and low-derivative expansion underlying vacuum polarization effects.

On the contrary, nonlocal terms arise as a contribution of the upper limit in the proper-time integral (1.10), which makes the late time asymptotics of $\mathrm{Tr}\,K(s)$ most important for another class of effects including particle creation and scattering[1]. Its integrand – the heat kernel trace, including its late time asymptotics, was first studied within the covariant nonlocal curvature expansion in [9, 11, 12, 13]. The goal of this paper is to give a critical overview of these old results and present the latest development in heat kernel theory concerning its nonperturbative asymptotics in flat spacetime [14], its generalization to curved spacetime geometry [15] and application in the theory of nonlocal effective action.

[1]Here the effective action is defined in Euclidean space with positive-signature metric. Its application in physical spacetime with Lorentzian signature is based on analytic continuation methods which range from a conventional Wick rotation in scattering theory (for in-out matrix elements) to a special retardation prescription in a wide class of problems for a mean field (in-in expectation value) [9, 10]. These methods nontrivially apply to nonlocal terms and extend from the usual perturbation theory to its partial resummation corresponding to the nonperturbative technique of the present work.

2. Approximation schemes and infrared behavior

To simplify the presentation, throughout this section we will work in flat spacetime with metric $g_{\mu\nu} = \delta_{\mu\nu}$ and, where necessary, briefly mention relevant modifications due to spacetime curvature. Also we will consider the case of a scalar field without spin, when the heat kernel (1.7) is a biscalar object without spin labels. In this case the small-time behavior of the heat kernel, which actually accounts for the success of renormalization scheme in local quantum field theory, looks especially simple and reads

$$K(s|\,x,y) = \frac{1}{(4\pi s)^{d/2}}\,\exp\left\{-\frac{|x-y|^2}{4s}\right\}\Omega(s|\,x,y) \tag{2.1}$$

$$\Omega(s|\,x,y) \to 1, \quad s \to 0, \tag{2.2}$$

where d is the spacetime dimensionality. This semiclassical ansatz for the heat kernel guarantees that at $s \to 0$ it tends to the delta-function $\delta(x,y)$ and contains all nontrivial information about the potential $V(x)$ in the function $\Omega(s|\,x,y)$ which is analytic at $s = 0$. Its expansion in powers of s underlies the technique of local Schwinger-DeWitt expansion which looks as follows.

2.1. Schwinger-DeWitt technique of local expansion

From the heat equation (1.8) one easily derives a set of recurrent equations for the coefficients of small-time expansion of $\Omega(s|\,x,y)$

$$\Omega(s|\,x,y) = \sum_{n=0}^{\infty} a_n(x,y)\,s^n, \quad s \to 0. \tag{2.3}$$

These coefficients play a very important role in quantum field theory and have the name of HAMIDEW coefficients that was coined by G.Gibbons to praize joint efforts of mathematicians and physicists in heat kernel theory and its implications. The equations for $a_n(x,y)$ can in closed form be solved for their coincidence limits $a_n(x,x)$ in terms of the potential $V(x)$ and its derivatives. For the operator (1.5)-(1.6) the first few of them read

$$\begin{aligned}
a_0(x,x) &= 1\\
a_1(x,x) &= -V(x) + \frac{1}{6}R(x)\\
a_2(x,x) &= \frac{1}{2}\,V^2(x) + \frac{1}{6}\,\Box V(x) - \frac{1}{6}\,R(x)V(x) + O(R^{\,2}_{\mu\nu\alpha\beta}),
\end{aligned} \tag{2.4}$$

where we also included the contribution of spacetime curvature scalar R and symbolically denoted the quadratic contribution of curvature tensor in the second-order coefficient. As is clearly seen, these quantities are local functions built of the coefficients of the original differential operator and their derivatives. The dimensionality of $a_n(x,x)$ in units of inverse length grows with n and is comprised of the powers of dimensionful quantities $V(x)$, $R_{\mu\nu\alpha\beta}$ and their derivatives.

Suppose now that instead of the theory with the operator (1.5) we consider the theory of massive field with a large mass m. This corresponds to the replacement of the original operator by

$$F(\nabla) \to F(\nabla) - m^2. \tag{2.5}$$

The corresponding heat kernel (2.1) under this replacement obviously acquires the overall exponential factor $\exp(-sm^2)$ damping the contribution of large values of s in the proper time integral (1.10)

$$K(s|\,x,y) = \frac{1}{(4\pi s)^{d/2}} \exp\left\{-\frac{|x-y|^2}{4s} - sm^2\right\} \Omega(s|\,x,y). \tag{2.6}$$

Substituting it together with the expansion (2.3) into (1.10) and integrating the resulting series term by term we obtain the one-loop effective action of massive theory in the form of the asymptotic $1/m^2$ expansion [11, 14, 16, 17]

$$\begin{aligned}\frac{1}{2}\,\mathrm{Tr}\,\ln\left[F(\nabla) - m^2\right] &= -\frac{1}{2\,(4\pi)^{d/2}}\int dx \int_0^\infty \frac{ds}{s^{d/2+1}}\, e^{-sm^2} \sum_{n=0}^{\infty} s^n\, a_n(x,x)\\ &= \mathit{\Gamma}_{\rm div} + \mathit{\Gamma}_{\rm log} - \frac{1}{2}\left(\frac{m^2}{4\pi}\right)^{d/2}\int dx \sum_{n=d/2+1}^{\infty} \frac{\Gamma(n-d/2)}{(m^2)^n}\, a_n(x,x).\end{aligned} \tag{2.7}$$

The first $d/2$ integrals (we assume that d is even) are divergent at the lower limit and generate ultraviolet divergences $\mathit{\Gamma}_{\rm div}$ given by the first $d/2$ Schwinger-DeWitt coefficients. In dimensional regularization they read

$$\mathit{\Gamma}_{\rm div} = \frac{1}{2(4\pi)^{d/2}}\int dx \sum_{n=0}^{d/2}\left[\frac{1}{\omega - d/2} - \Gamma'\Big(\frac{d}{2} - n + 1\Big)\right]\frac{(-m^2)^{d/2-n}}{(d/2-n)!}\, a_n(x,x). \tag{2.8}$$

The logarithmic divergences are also accompanied by the logarithmic term

$$\mathit{\Gamma}_{\rm log} = \frac{1}{2(4\pi)^{d/2}}\int dx \sum_{n=0}^{d/2} \frac{(-m^2)^{d/2-n}}{(d/2-n)!}\,\ln\frac{m^2}{\mu^2}\, a_n(x,x), \tag{2.9}$$

containing the renormalization mass parameter μ^2 reflecting the renormalization ambiguity.

In the present form each term in the finite part of the action (2.7) is local, but this locality holds only in the range of applicability of this expansion when the mass is large and the terms of the asymptotic series rapidly decrease with the growth of n. This occurs when the mass parameter m is large compared with *both* the derivatives of the potential and the potential itself

$$1 \gg \frac{a_n}{(m^2)^n} \sim \left(\frac{V}{m^2}\right)^n,\ \left(\frac{\nabla}{m}\right)^k\!\left(\frac{V}{m^2}\right)^l, \quad k + 2l = 2n\,. \tag{2.10}$$

In the presence of the gravitational field these restrictions include also the smallness of spacetime curvature and its covariant derivatives compared to the mass parameter.

Thus the local Scwinger-DeWitt expansion is applicable only for slow varying fields of small amplitude compared to the mass scale of the model.

For large field strengths or rapidly varying fields the Schwinger-DeWitt expansion becomes inapplicable and completely blows up in the massless limit $m \to 0$. So the question arises, what is the structure of the effective action in this situation and how to calculate it. Below we consider two perturbation methods which improve the Schwinger-DeWitt technique to extend it to the class of massless models and then, in Sect.3, go over to the nonperturbative technique based on the nonperturbative late-time asymptotics of the heat kernel.

2.2. Modified Schwinger-DeWitt expansion

The first method may be called the modified Schwinger-DeWitt expansion, because it is based on the resummation of this expansion originating from the replacement of the mass term by the potential $V(x)$ of the operator (1.7), $m^2 \to V(x)$. When the potential is positive-definite (which we shall assume here) it can play the role of the damping factor similar to the mass term in the integral (2.7). For this purpose we have to extract the exponential dependence on $sV(x)$ from the function $\Omega(s|\,x,y)$, $\Omega(s|\,x,y) = e^{-sV(x)}\tilde{\Omega}(s|\,x,y)$, and write instead of (2.6) the ansatz

$$K(s|\,x,y) = \frac{1}{(4\pi s)^{d/2}}\,\exp\left\{-\frac{|x-y|^2}{4s} - V(x)\right\}\tilde{\Omega}(s|\,x,y), \tag{2.11}$$

where the new function has the expansion in s with the modified Schwinger-DeWitt coefficients

$$\tilde{\Omega}(s|\,x,y) = \sum_{n=0}^{\infty} \tilde{a}_n(x,y)\, s^n. \tag{2.12}$$

Obviously this expansion represents the partial resummation of the initial series (2.3) in powers of the undifferentiated potential. New coefficients $\tilde{a}_n(x,x)$ in contrast to old ones have fewer number of terms and do not contain undifferentiated potential (and, therefore, except for $\tilde{a}_0 = 1$ vanish in the absence of gravity for $\nabla V = 0$). The original coefficients can be expressed in terms of them as finite-order polynomials in V with coefficients built of the gradients of potential

$$a_m(x,x) = \frac{(-V)^m}{m!} + \sum_{n=1}^{d/2} \frac{(-V)^{m-n}}{(m-n)!}\,\tilde{a}_n(x,x) = \frac{(-V)^m}{m!} + O(\nabla V). \tag{2.13}$$

Now the proper time integral in (2.7) even for $m^2 = 0$ has an infrared cutoff at $s \sim 1/V(x)$ and in this case the effective action is similar to (2.7), where m^2 is replaced by $V(x)$ and $a_n(x,x)$ by $\tilde{a}_n(x,x)$

$$\sum_n \frac{a_n(x,x)}{(m^2)^n} \to \sum_n \frac{\tilde{a}_n(x,x)}{V^n(x)}. \tag{2.14}$$

In particular, the ultraviolet divergences are given by the massless limit of $\varGamma_{\rm div}$ and the logarithmic part gives rise to the (d-dimensional) Coleman-Weinberg term $\varGamma_{\log} \to \varGamma_{\rm CW} + O(\nabla V)$,

$$\varGamma_{\rm CW} = \frac{1}{2(4\pi)^{d/2}} \int dx\, \frac{(-V)^{d/2}(x)}{(d/2)!} \ln \frac{V(x)}{\mu^2}, \tag{2.15}$$

corrected by the contribution due to the derivatives of $V(x)$ (confer Eq.(2.13) with $m = d/2$). $\varGamma_{\rm CW}$ here is the spacetime integral of the Coleman-Weinberg effective potential. For instance, in four dimensions in the φ^4-model of the self-interacting scalar field with $V(\varphi) \sim \varphi^2$, this is the original Coleman-Weinberg effective potential, $\varphi^4 \ln(\varphi^2/\mu^2)/64\pi^2$.

From (2.14) it follows that, in contrast to the original Schwinger-DeWitt series, its modified version represents the expansion in the derivatives of V rather than powers of V itself. Indeed, the modified Schwinger-DeWitt coefficients do not contain the undifferentiated potential and the typical structure of the terms entering $\tilde{a}_n(x,x)$ is represented by m derivatives acting in different ways on the product of j potentials, $\nabla^m V^j(x)$, where $m + 2j = 2n$. Every V here should be differentiated at least once and therefore $m \geq j$. Thus the typical terms of the expansion (2.14) can be symbolically written down as

$$\frac{\tilde{a}_n}{V^n} \sim \sum_{j=1}^{[2n/3]} \frac{\nabla^{2n-2j} V^j}{V^n}, \tag{2.16}$$

where the upper value of j is the integer part of $2n/3$. Therefore this expansion is efficient as long as the potential is slowly varying in units of the potential itself

$$\frac{\nabla^2 V(x)}{V^2(x)} \ll 1, \quad \frac{\left(\nabla V(x)\right)^2}{V^3(x)} \ll 1, \quad \dots . \tag{2.17}$$

When the potential is bounded from below by a large positive constant this condition can be easily satisfied throughout the whole spacetime. But this case is uninteresting because it reproduces the original Schwinger-DeWitt expansion with m^2 playing the role of this bound. More interesting is the case of the asymptotically empty spacetime when the potential and its derivatives fall off to zero by the power law

$$V(x) \sim \frac{1}{|x|^p}, \quad \nabla^m V(x) \sim \frac{1}{|x|^{p+m}}, \quad |x| \to \infty \tag{2.18}$$

for some positive p. For such a potential terms of the perturbation series (2.14) behave as

$$\frac{\tilde{a}_n(x,x)}{V^n(x)} \sim \sum_{j=1}^{[2n/3]} |x|^{(p-2)(n-j)} \tag{2.19}$$

and thus decrease with n only if $p < 2$. For $p \geq 2$ the modified gradient expansion completely breaks down. It makes sense only for slowly decreasing potentials of the

form (2.18) with $p < 2$. In this case the potential $V(x)$ is not integrable over the whole spacetime $\left(\int dx\, V(x) = \infty\right)$ and moreover even the operation $(1/\Box)V(x)$ is not well defined[2]. Therefore, the above restriction is too strong to account for interesting physical problems in which the parameter p typically coincides with the spacetime dimensionality d. In addition, the modified asymptotic expansion is completely local and does not allow one to capture nonlocal terms of effective action.

Thus an alternative technique is needed to obtain the late-time contribution to the proper-time integral and, in particular, to understand whether and when this integral exists at all in massless theories. The answer to this question lies in the late-time asymptotics of the heat kernel at $s \to \infty$ which can be perturbatively analyzed within the covariant perturbation theory of [9, 11, 12, 13, 18].

2.3. Covariant perturbation theory

In the covariant perturbation theory the full potential $V(x)$ is treated as a perturbation and the solution of the heat equation is found as a series in its powers. From the viewpoint of the Schwinger-DeWitt expansion it corresponds to an infinite resummation of all terms with a given power of the potential and arbitrary number of derivatives. The result reads as

$$\mathrm{Tr}\, K(s) \equiv \int dx\, K(s|\, x, x) = \sum_{n=0}^{\infty} \mathrm{Tr}\, K_n(s), \tag{2.20}$$

where

$$\mathrm{Tr}\, K_n(s) = \int dx_1 dx_2 ... dx_n\, F_n(s|\, x_1, x_2, ... x_n)\, V(x_1) V(x_2) ... V(x_n), \tag{2.21}$$

and the nonlocal form factors $F_n(s|\, x_1, x_2, ... x_n)$ were explicitly obtained in [9, 11, 12] up to $n = 3$ inclusive. In the presence of gauge fields and gravity this expansion can be easily generalized by including the fibre bundle $\mathcal{R}_{\mu\nu}$ and Ricci curvature[3] in the full set of perturbatively treated field strengths, $V \to \mathcal{R} = (V, \mathcal{R}_{\mu\nu}, R_{\mu\nu})$ and covariantizing the corresponding nonlocal form factors.

It was shown [11] that at $s \to \infty$ the terms in this expansion behave as

$$\mathrm{Tr}\, K_n(s) = O\left(\frac{1}{s^{d/2-1}}\right), \quad n \geq 1, \tag{2.22}$$

and, therefore in spacetime dimension $d \geq 3$ the integral in (1.10) is infrared convergent

$$\varGamma \sim \int^{\infty} \frac{ds}{s}\, O\left(\frac{1}{s^{d/2-1}}\right) < \infty. \tag{2.23}$$

[2] For the convergence of the integral in $(1/\Box)\, V$ the potential $V(x)$ should fall off at least as $1/|x|^3$ in any spacetime dimension [11].

[3] In asymptotically flat spacetime with natural vacuum boundary conditions the Riemannian curvature can be perturbatively expressed in terms of the Ricci tensor [12, 18], that is why it does not enter the expansion as an independent entity.

In one and two dimensions this expansion for Γ does not exist except for the special case of the massless theory in curved two-dimensional spacetime, when it reproduces the Polyakov action [11, 19, 20], which alternatively can be obtained by integrating the conformal anomaly [19, 20].

$$\Gamma_{\rm Polyakov} \sim \int d^2x\, g^{1/2}\, R\frac{1}{\Box}R. \tag{2.24}$$

Covariant perturbation theory should always be applicable whenever $d \geq 3$ and the potential V is sufficiently small, so that its effective action explicitly features analyticity in the potential at $V = 0$. Therefore, its serious disadvantage is that this theory does not allow one to overstep the limits of perturbation scheme and, in particular, discover non-analytic structures in the action if they exist.

3. Nonperturbative late-time asymptotics

Nonperturbative technique for the heat kernel is based on the approximation qualitatively different from those of the previous section. Rather than imposing certain smallness restrictions on the background fields we consider them rather generic, but consider the limit of large proper time $s \to \infty$. Continuing working in flat spacetime with $g_{\mu\nu} = \delta_{\mu\nu}$, we substitute the ansatz (2.1) in the heat equation and immediately obtain the following equation for the unknown function $\Omega(s|\,x, y)$

$$\frac{\partial\Omega}{\partial s} + \frac{1}{s}\,(x-y)^\mu\nabla_\mu\Omega = F(\nabla)\,\Omega. \tag{3.1}$$

Then we assume the existence of the following $1/s$-expansion for this function (which follows, in particular, from the perturbation theory for $K(s|\,x, y)$ [11, 14] briefly reviewed above – there is no nonanalytic terms in $1/s$ like $\ln(1/s)$),

$$\Omega(s|\,x, y) = \Omega_0(x, y) + \frac{1}{s}\,\Omega_1(x, y) + O\left(\frac{1}{s^2}\right) \tag{3.2}$$

and obtain the series of recurrent equations for the coefficients of this expansion

$$F(\nabla)\,\Omega_0(x, y) = 0, \tag{3.3}$$
$$F(\nabla)\,\Omega_1(x, y) = (x-y)^\mu\nabla_\mu\Omega_0(x, y), \tag{3.4}$$
$$...$$

An obvious difficulty with the choice of the concrete solution for this chain of equations is that they do not form a well posed boundary value problem. Point is that natural zero boundary conditions at spacetime infinity for the original kernel $K(s|\,x, y)$ do not impose any boundary conditions on the function $\Omega(s|\,x, y)$ except maybe the restriction on the growth of $\Omega(s|\,x, y)$ to be slower than $\exp\,[+|x-y|^2/2s]$, because of the exponential factor in (2.1). On the other hand, this freedom in choosing non-decreasing at $|x| \to \infty$ solutions essentially helps to find them, because in the opposite case even the existence of nontrivial solutions would be violated. Indeed the elliptic equation

(3.3) with positive definite operator $F(\nabla)$ (which we assume) would not have nonzero solutions decaying at spacetime infinity. Thus, the only remaining criterion for the selection of solutions in (3.3)-(3.4) is the requirement of symmetry of the coefficients $(\Omega_0(x,y),\,\Omega_1(x,y),...)$ in their arguments. As we will see now, this criterion taken together with certain assumptions of *naturalness* result in concrete solutions which will be further checked on consistency by different methods including perturbation theory, the variational equation for the heat kernel trace and its metric analogue, etc.

The absence of falloff properties for the coefficients of the expansion (3.2) results in one more interesting peculiarity of the late-time expansion. As we will now see, this expansion for the functional *trace* of the heat kernel corresponding to (3.2) reads

$$\mathrm{Tr}\,K(s) = \frac{1}{(4\pi s)^{d/2}}\,\left\{ s\,W_0 + W_1 + O\left(\frac{1}{s}\right)\right\}, \tag{3.5}$$

which obviously implies that in spite of (1.11) $W_n \neq \int dx\,\Omega_n(x,x)$, $n = 0, 1, ...$, because of the unit shift in the power of s. The explanation of this visible mismatch between (3.2) and (3.5) lies in the observation that the $1/s$-expansion (3.2) is not uniform in x and y arguments of $\Omega(s|\,x,y)$. Therefore, for fixed s the expression $\Omega(s|\,x,x)$ taken from this expansion fails to be correct for $|x| \to \infty$ and, as a consequence, the heat kernel functional trace (requiring integration up to spacetime infinity) cannot be obtained by applying (1.11) to (3.2).

Fortunately, there is a means of circumventing this difficulty. The functional trace of the heat kernel can be recovered from its expansion due to the following variational equation

$$\frac{\delta\,\mathrm{Tr}\,K(s)}{\delta V(x)} = -sK(s|x,x), \tag{3.6}$$

which is a direct corollary of the heat kernel definition. One power of s on the right hand side of this equation explains, in particular, extra power of the proper time in (3.5) as compared to (3.2). This equation reduces to the sequence of variational equations

$$\frac{\delta\,W_n}{\delta V(x)} = -\Omega_n(x,x), \qquad n = 0, 1, ...\,, \tag{3.7}$$

which will be used to obtain the first two coefficients in (3.5).

3.1. Leading order

The way the strategy outlined above works in the leading order of the $1/s$-expansion is as follows [14]. Make a natural *assumption* that $\Omega_0(x,y)$ at $|x| \to \infty$ is not growing and *independent* of the angular direction $n^\mu = x^\mu/|x|$ quantity $C(y)$ – the function of only y. Then the solution of Eq.(3.3) subject to boundary condition $\Omega_0(x,y)\,|_{|x|\to\infty} = C(y)$, is unique and reads $\Omega_0(x,y) = \Phi(x)\,C(y)$, where $\Phi(x)$ is a special function

$$\Phi(x) = 1 + \frac{1}{\Box - V}\,V(x) \equiv 1 + \int dy\,G(x,y)V(y) \tag{3.8}$$

solving the homogeneous equation subject to unit boundary conditions at infinity

$$\begin{cases} F(\nabla)\,\Phi(x) = 0, \\ \\ \Phi(x) \to 1, \quad |x| \to \infty. \end{cases} \tag{3.9}$$

Then, the requirement of symmetry in x and y implies that $\Omega_0(x,y) = C\,\Phi(x)\,\Phi(y)$, where the value of the numerical normalization coefficient $C\!=\!1$ follows from the comparison with the exactly known heat kernel in flat spacetime with vanishing potential $V(x) = 0$. Thus

$$\Omega_0(x,y) = \Phi(x)\,\Phi(y). \tag{3.10}$$

Note that with this result treated as valid up to infinity in x, the heat kernel trace becomes badly defined because the integral of the coincidence limit of $\Omega_0(x,y)$, is divergent

$$\int dx\,\Omega(s|x,x) = \int dx\,\Phi^2(x) = \infty.$$

On the contrary, the variational equation (3.7) for W_0 satisfies the integrability condition in view of the relation

$$\frac{\delta\Phi(x)}{\delta V(y)} = G(x,y)\,\Phi(y) \tag{3.11}$$

and has the following solution in the form of a well-defined convergent integral [14]

$$W_0 = -\int dx\,V\,\Phi(x). \tag{3.12}$$

3.2. Subleading order

In the subleading order the equation (3.4) takes the form

$$F(\nabla)\,\Omega_1(x,y) = (x-y)^\mu\nabla_\mu\Phi(x)\Phi(y), \tag{3.13}$$

with the inhomogeneous term on the right hand side, which is slowly tending to zero at infinity in $|x|$ (and growing in $|y|$). A symmetric in x and y solution that was found in [15]

$$\begin{aligned}\Omega_1(x,y) = {} & \frac{1}{\Box_x - V_x}\,(x-y)^\mu\nabla_\mu\Phi(x)\,\Phi(y) + (x\leftrightarrow y)\\ & + 2\,\frac{1}{\Box_x - V_x}\nabla_\mu\Phi(x)\,\frac{1}{\Box_y - V_y}\nabla^\mu\Phi(y),\end{aligned} \tag{3.14}$$

has linearly growing in x and y terms[4], which is a corollary of the missing falloff property for the right hand side of Eq.(3.13). Thus, this solution is essentially non-unique,

[4]Here the label of the differential operator in the denominator indicates on which argument the corresponding Green's function is acting.

but the correctness of its choice can be supported by the fact that the corresponding coincidence limit $\Omega_1(x,x)$ guarantees the integrability condition of the variational equation (3.7) for W_1 [15] and confirmed by the direct summation of the perturbative series for $\mathrm{Tr}\,K(s)$ (see below and Appendix A). The answer for W_1 reads

$$W_1 = \int dx \left\{ 1 - 2\,\nabla_\mu\Phi\,\frac{1}{\Box - V}\,\nabla^\mu\Phi \right\}, \tag{3.15}$$

where the nontrivial nonlocal term quadratic in gradients of $\Phi(x)$ is given by a well-defined convergent integral in distinction from the unit term which was added as a functional integration constant following from the comparison of this result with the exactly known case of a vanishing potential.

3.3. Resummation of covariant perturbation theory

To substantiate the nonperturbative algorithms of this section and, in particular, to check the correct choice of solutions for the coefficients of the $1/s$-expansion (3.2) one can compare them with the result of the resummation of nonlocal series in the perturbation technique of [9, 11, 12]. In this technique the functional trace of the heat kernel is expanded as nonlocal series (2.20)-(2.21) in powers of the potential V with explicitly calculable coefficients - nonlocal form factors $F_n(s|\,x_1, x_2, ...x_n)$. Their leading asymptotic behavior at large s was obtained in [11], and it can also be extended to the first subleading order in $1/s$ [14]. Then in this approximation one can explicitly perform infinite summation of power series in the potential to confirm the answers for W_0 and W_1 obtained above.

According to [11] the heat kernel trace is local in the first two orders of the perturbation theory (2.20)

$$\mathrm{Tr}\,K_0(s) = \frac{1}{(4\pi s)^{d/2}} \int dx, \tag{3.16}$$

$$\mathrm{Tr}\,K_1(s) = -\frac{s}{(4\pi s)^{d/2}} \int dx\, V(x), \tag{3.17}$$

and in higher orders it reads as

$$\mathrm{Tr}\,K_n(s) = \frac{(-s)^n}{(4\pi s)^{d/2} n} \int dx \left\langle e^{s\Omega_n} \right\rangle V(x_1)V(x_2)...V(x_n)\Big|_{x_1=...=x_n=x}, \quad n \geq 2. \tag{3.18}$$

Here Ω_n is a differential operator acting on the product of n potentials

$$\Omega_n = \sum_{i=1}^{n-1} \nabla_{i+1}^2 + 2 \sum_{i=2}^{n-1} \sum_{k=1}^{i-1} \beta_i(1-\beta_k)\nabla_{i+1}\nabla_{k+1}, \tag{3.19}$$

expressed in terms of the partial derivatives labelled by the indices i implying that ∇_i acts on $V(x_i)$. It is assumed in (3.18) that after the action of all derivatives on the respective terms all x_i are set equal to x. It is also assumed that the spacetime indices of all derivatives $\nabla = \nabla^\mu$ are contracted in their bilinear combinations, $\nabla_i\nabla_k \equiv \nabla_i^\mu \nabla_{\mu\,k}$.

The differential operator (3.19) depends on the parameters $\beta_i,\ i=1,...,n-1$, which are defined in terms of the parameters $\alpha_i,\ i=1,...,n$ as

$$\beta_i = \alpha_{i+1} + \alpha_{i+2} + ... + \alpha_n, \tag{3.20}$$

and the angular brackets in $\langle e^{s\Omega_n}\rangle$ imply that this operator exponent is integrated over compact domain in the space of α-parameters

$$\langle e^{s\Omega_n}\rangle \equiv \int\limits_{\alpha_i\geq 0} d^n\alpha\, \delta\Big(\sum_{i=1}^n \alpha_i - 1\Big)\ \exp(s\Omega_n). \tag{3.21}$$

The late time behavior of $\mathrm{Tr}\, K_n(s)$ is thus determined by the asymptotic behavior of this integral at $s\to\infty$, which can be calculated using the Laplace method. To apply this method, let us note that Ω_n is a *negative* semidefinite operator (this is shown in the Appendix B of [11]) which degenerates to zero at n points of the integration domain: $(0,...,0,\alpha_i=1,0,...,0),\ i=1,...,n$. Therefore the asymptotic expansion of this integral is given by the contribution of the corresponding n maxima of the integrand at these points. The integration by parts in (3.18) is justified by the formal identity $\nabla_1+\nabla_2+...+\nabla_n=0$. Using it one can show that the contributions of all these maxima are equal, so that it is sufficient to calculate only the contribution of the point $\alpha_1=1,\ \alpha_i=0,\ i=2,...,n$. In the vicinity of this point it is convenient to rewrite the expression for Ω_n in terms of the independent $(n-1)$ variables $\alpha_2,\alpha_3,...,\alpha_n$, the remaining $\alpha_1 = 1-\sum_{i=2}^n\alpha_i$,

$$\Omega_n = \sum_{i=2}^n \alpha_i D_i^2 - \sum_{m,k=2}^n \alpha_m\alpha_k D_m D_k, \tag{3.22}$$

where the operator D_m is defined as

$$D_m = \nabla_2 + \nabla_3 + ... + \nabla_m, \quad m=2,...,n. \tag{3.23}$$

The details of the further calculation are presented in the Appendix A. Here we only specify the structure of the result in the n-th order of perturbation theory, which shows that one can perform explicit summation of nonlocal perturbation series leading to (3.12) and (3.15). This structure is most characteristic in the leading order approximation. The n-th order term reads

$$\begin{aligned}\mathrm{Tr}\ K_n(s) &= \frac{1}{(4\pi s)^{d/2}}\int dx\ \left[-s\frac{1}{D_2^2...D_n^2} + O\left(s^0\right)\right]\ V_1V_2...V_n\\ &= -\frac{s}{(4\pi s)^{d/2}}\int dx\ \underbrace{V\frac{1}{\Box}V\frac{1}{\Box}...V\frac{1}{\Box}}_{n-1}\ V(x) + O\left(\frac{1}{s^{d/2}}\right),\end{aligned} \tag{3.24}$$

where in view of the definition of the generalized derivative (3.23) the action of the nonlocal operator $1/D_2^2...D_n^2$ on the multi-point product $V_1V_2...V_n$ is rewritten as the $(n-1)$-th power of the nonlocal operator $V\times 1/\Box$ *acting to the right* on the first power

of V. Thus the expression (3.24) turns out to be the term of geometric progression in powers of $V(1/\Box)$, and the summation yields

$$\begin{aligned}\mathrm{Tr}\, K(s) &= \frac{1}{(4\pi s)^{d/2}} \int dx \left\{ -s\,\Box \frac{1}{\Box - V}\, V(x) + O(s^0) \right\} \\ &= \frac{1}{(4\pi s)^{d/2}} \int dx \left\{ -s\, V\Phi(x) + O(s^0) \right\}. \end{aligned} \tag{3.25}$$

Similarly, in the subleading order the calculation reduces to the summation of multiple – duplicate and triplicate – geometric progressions in powers of the same nonlocal operator. This summation gives

$$\mathrm{Tr}\, K(s) = \frac{1}{(4\pi s)^{d/2}} \int dx \left\{ -sV\Phi + 1 - 2\nabla_\mu\Phi \frac{1}{\Box - V} \nabla^\mu \Phi + O\left(s^{-1}\right) \right\} \tag{3.26}$$

and, thus, fully confirms Eqs.(3.12) and (3.15) obtained by another essentially nonperturbative method.

4. New nonlocal effective action

Nonperturbative asymptotics of the heat kernel allows one to improve essentially the calculation of the effective action. The new approximation which unifies the knowledge of both the early-time and late-time behaviors of $\mathrm{Tr}\, K(s)$ incorporates both the ultraviolet and infrared properties of the theory and generates new structures in the effective action. These structures generalize the Coleman-Weinberg local potential (2.15) to physical situations when the field is not homogeneous throughout the whole spacetime, but rather tends to zero at infinity. As we saw in Sect.2.2 only for extremely slow and physically uninteresting falloff (2.18) with $p < 2$ the deviation from homogeneity can be treated by perturbations. For a faster decrease at $|x| \to \infty$ the infrared cutoff of the proper-time integral fails within the modified gradient expansion, and we have to use either the nonlocal perturbation theory of Sect.2.3 or the nonperturbative asymptotics of Sect.3. The latter allows one to obtain both nonlocal and nonperturbative action which captures in a nontrivial way the edge effects of a transition domain between the spacetime interior at finite $|x|$ to vanishing potential at $|x| \to \infty$.

The key idea to build this new approximation is to replace $\mathrm{Tr}\, K(s)$ in (1.10) by some approximate function $\mathrm{Tr}\, \bar{K}(s)$ such that both early and late time asymptotics are satisfied and the integral over s is explicitly calculable. Here we exploit the simplest possibility – namely, take two simple functions $\mathrm{Tr}\, K_<(s)$ and $\mathrm{Tr}\, K_>(s)$

$$\mathrm{Tr}\, K_<(s) = \frac{1}{(4\pi s)^2} \int dx\, e^{-sV}, \tag{4.1}$$

$$\mathrm{Tr}\, K_>(s) = \frac{1}{(4\pi s)^2} \int dx\, (1 - sV\Phi), \tag{4.2}$$

which coincide with the leading asymptotics of $\mathrm{Tr}\, K(s)$ at $s \to 0$ (modified gradient expansion with derivative terms omitted) and $s \to \infty$ and use them to approximate

$\mathrm{Tr}\, K(s)$ respectively at $0 \le s \le s_*$ and $s_* \le s < \infty$ for some s_*

$$\mathrm{Tr}\,\bar{K}(s) = \begin{cases} \mathrm{Tr}\, K_<(s), & s < s_*, \\ \\ \mathrm{Tr}\, K_>(s), & s > s_*. \end{cases} \tag{4.3}$$

The value of s_* will be determined from the requirement that these two functions match at s_*, which will guarantee the stationarity of $\bar{\varGamma}$ with respect to the choice of s_*, $\partial\bar{\varGamma}/\partial s_* = 0$,

$$\mathrm{Tr}\, K_<(s_*) = \mathrm{Tr}\, K_>(s_*). \tag{4.4}$$

Thus, the new approximation for the action reads as

$$\bar{\varGamma} = -\frac{1}{2}\int\limits_0^{s_*} \frac{ds}{s}\mathrm{Tr}\, K_<(s) - \frac{1}{2}\int\limits_{s_*}^{\infty} \frac{ds}{s}\mathrm{Tr}\, K_>(s), \tag{4.5}$$

and its deviation from the exact $\varGamma_{1-\mathrm{loop}}$ proportional to $\mathrm{Tr}\,[K(s) - \bar{K}(s)]$ can then be treated by perturbations. This piecewise-smooth approximation is efficient only if the ranges of validity of two asymptotic expansions (respectively for small and big s) overlap with each other and the point s_* belongs to this overlap. In this case the corrections due to the deviation of $\mathrm{Tr}\,\bar{K}(s)$ from the exact $\mathrm{Tr}\, K(s)$ are uniformly bounded everywhere and one can expect that (4.5) would give a good zeroth-order approximation to an exact result. This requirement can be satisfied at least for two rather wide classes of potentials $V(x)$. They have finite amplitude V_0 within their compact support of size R [14],

$$\begin{aligned} &V(x) = 0, \quad |x| \ge R, \\ &V(x) \sim V_0, \quad |x| \le R, \end{aligned} \tag{4.6}$$

and have the property that their derivatives are not too high and uniformly bounded by the quantity of the order of magnitude V_0/R.

One class is when the potential is small in units of the inverse size of its compact support

$$V_0 R^2 \ll 1. \tag{4.7}$$

Consider first the case of four-dimensional spacetime, $d = 4$. Simple calculation presented in Appendix B yields in this case the following answer [14] for the finite part of the action, which is valid up to corrections proportional to this smallness parameter

$$\begin{aligned} \bar{\varGamma} \simeq \frac{1}{64\pi^2}\int d^4x\, V^2(x)\, \ln\Big(\int d^4y\, V^2(y)\Big) \\ -\frac{1}{64\pi^2}\int d^4x\, V^2(x) \ln\Big(\int d^4y\, V\frac{\mu^2}{V - \Box}V(y)\Big)\,, \quad d = 4. \end{aligned} \tag{4.8}$$

In what follows we disregard the ultraviolet divergent part of the action and absorb all finite renormalization type terms $\sim \int dx\, V^{d/2}$ in the redefinitions of μ^2.

Note, that this renormalization mass parameter μ^2 makes the argument of the second logarithm dimensionless and plays the same role as for the Coleman-Weinberg potential. However, the original Coleman-Weinberg term for small potentials of the type (4.7) gets replaced by the other qualitatively new nonlocal structure. For small potentials spacetime gradients dominate over their magnitude and, therefore, the Coleman-Weinberg term does not survive in this approximation. Still it can be recovered in the formal limit of the constant potential, when the argument of the second logarithms tends to $\mu^2 \int dx\, V$ and the infinite volume factor ($\int dx$) gets cancelled in the difference of two logarithms,

$$\bar{\varGamma} \to \varGamma_{\rm CW} \equiv \frac{1}{64\pi^2} \int d^4x\, V^2 \ln \frac{V}{\mu^2}, \quad V(x) \to {\rm const.} \tag{4.9}$$

This justifies the consistency of this approximation.

As shown in Appendix B, for higher (even) dimensions $d > 4$ the logarithmic term of the action turns out to be subleading, and the answer is dominated by the following renormalization-independent and *negative-definite* part of the expression

$$\begin{aligned}\bar{\varGamma} \simeq -\frac{1}{(8\pi)^{d/2}} \frac{2}{d\,(d-2)} \int dx\, V^2(x) \left(\frac{\int dy\; V^2(y)}{\int dy\, V \frac{1}{V-\Box} V(y)} \right)^{d/2-2} \\ + \frac{1}{2\,(4\pi)^{d/2}} \int dx\, \frac{(-V)^{d/2}(x)}{(d/2)!}\, \ln \left(\frac{\int dy\; V^2(y)}{\int dy\, V \frac{\mu^2}{V-\Box} V(y)} \right).\end{aligned} \tag{4.10}$$

The second term here for small but spatially variable potential is generally much smaller than the first one, because within the bound (4.7) (see Appendix B)

$$\frac{\int dy\; V^2(y)}{\int dy\, V \frac{1}{V-\Box} V(y)} \gg V(x)\,. \tag{4.11}$$

However, in the limit of constant potential this ratio (which is actually $1/s_*$) tends to V, so that both terms become of the same order of magnitude. The first term then turns out to be of a purely renormalization nature $\sim \int dx\, V^{d/2}$, while the second term goes over into the (logarithmically stronger) d-dimensional Coleman-Weinberg potential (2.15). So the correspondence principle holds also in higher dimensions.

Another class of potentials, when the piecewise smooth approximation is effective, corresponds to the opposite limit,

$$V_0 R^2 \gg 1, \tag{4.12}$$

that is big potentials in units of the inverse size of their support. In this case spacetime gradients do not dominate the amplitude of the potential and the calculation in Appendix B shows that the effective action contains the Coleman-Weinberg term modified by the special nonlocal correction (previously derived for $d = 4$ in [14])

$$\bar{\varGamma} \simeq \varGamma_{\rm CW} + \frac{1}{(4\pi)^{d/2}} \frac{2}{d\,(d-2)} \int\limits_{|x| \leq R} dx\, \big\langle V\Phi \big\rangle^{d/2}. \tag{4.13}$$

This correction involves the average value of the function $V\Phi(x)$ on the compact support domain

$$\big\langle V\Phi\big\rangle \equiv \frac{\int\limits_{|x|\leq R} dx\, V\Phi}{\int\limits_{|x|\leq R} dx}. \tag{4.14}$$

Again this algorithm correctly stands the formal limit of a constant potential, because in this limit the function $\Phi(x)$ given by (3.8) formally tends to zero (and the size R grows to infinity).

5. Inclusion of gravity

Remarkable property of the obtained late-time asymptotics is that it can be nearly straightforwardly generalized to curved spacetime. In this case the flat metric gets replaced by the curved one and the interval in the ansatz (2.1) goes over to the world function – one half of the geodesic distance squared between the points x and y

$$\begin{aligned} &\delta_{\mu\nu} \to g_{\mu\nu}(x), \\ &\frac{|x-y|^2}{2} \to \sigma(x,y). \end{aligned} \tag{5.1}$$

The semiclassical ansatz (2.1) takes the form

$$K(s|\,x,y) = \frac{1}{(4\pi s)^{d/2}}\, \exp\left[-\frac{\sigma(x,y)}{2s}\right]\, \Omega(s|\,x,y)\, g^{1/2}(y), \tag{5.2}$$

where $\Omega(s|\,x,y)$ is a biscalar quantity which instead of (2.3) has a small-time limit

$$\begin{aligned} &\Omega(s|\,x,y) \;\to\; \Delta^{1/2}(x,y), \quad s \to 0, \\ &\Delta(x,y) = g^{-1/2}(x)\left(\det \partial^x_\mu \partial^y_\nu \sigma(x,y)\right) g^{-1/2}(y) \neq 0 \end{aligned} \tag{5.3}$$

in terms of the (dedensitized) Pauli-Van Vleck-Morette determinant [16, 17][5].

Disentangling of $\Delta^{1/2}(x,y)$ as a separate factor in (5.1) is not useful for the purposes of late time expansion. However, the quantity is rather important and related to a serious simplifying assumption which underlies our results. The assumption we make is the absence of focal points in the congruence of geodesics determining the world function $\sigma(x,y)$. We assume that for all pairs of points x and y, $\Delta(x,y) \neq 0$, which guarantees that $\sigma(x,y)$ is globally and uniquely defined on the asymptotically-flat spacetime in question. This assumption justifies the ansatz (5.2) which should be globally valid because the coefficients of the expansion (3.2) will satisfy elliptic boundary-value problems with boundary conditions at infinity.

[5] We define the $\delta(x,y)$-function as a scalar with respect to the first argument x and as a density of unit weight with respect to the second one – y. Correspondingly the heat kernel has the same weights of their arguments. This asymmetry in x and y explains the presence of the factor $g^{1/2}(y)$ in (5.2) and a biscalar nature of $\Omega(s|\,x,y)$.

We also assume that the metric is asymptotically flat and has at spacetime infinity in cartesian coordinates the following falloff behavior characteristic of d-dimensional Euclidean spacetime

$$g_{\mu\nu}(x)\Big|_{|x|\to\infty}=\delta_{\mu\nu}+O\left(\frac{1}{|x|^{d-2}}\right). \tag{5.4}$$

This requirement does not exclude caustics in the geodesic flow, $\Delta(x,y)=0$, which depend on local properties of the gravitational field, unrelated to its long-distance behavior. The assumption of geodesic convexity might be too strong to incorporate physically interesting situations, but we believe that the late-time asymptotics will survive in the presence of caustics (though, maybe by the price of additional contributions which go beyond the scope of this paper)[6].

5.1. Leading order

Repeating the arguments of Sect.3 one immediately finds the sequence of recurrent equations for the coefficients of the $1/s$-expansion (3.2) for the operator $F(\nabla)$, (1.5)-(1.6), in curved spacetime. Their solution in the leading order turns out to be a straightforward covariantization of the flat-space result (3.10) with the universal scalar function $\Phi(x)$ being now a functional of both metric and potential,

$$\Phi(x)\equiv\Phi(x)[g_{\mu\nu},V]=1+\int dy\,G(x,y)V(y), \tag{5.5}$$

in terms of the curved-space Green's function $G(x,y)\equiv G(x,y)[V,g_{\mu\nu}]$ subject to the same Dirichlet boundary conditions at infinity

$$\begin{cases}F(\nabla)\,G(x,y)=\delta(x,y),\\ G(x,y)\big|_{|x|\to\infty}=0\end{cases} \tag{5.6}$$

Thus, the leading order of late-time expansion for $K(s|\,x,y)$ is just a direct covariantization of the flat-space result. Remarkably, almost the same situation holds for the functional trace. Its leading order is given by two terms. One of them is a straightforward covariantization of (3.13) and another represents a new structure – the surface integral over spacetime infinity reflecting the asymptotically-flat properties of its metric.

$$W_0=-\int dx\,g^{1/2}\,V\,\Phi(x)+\frac{1}{6}\,\Sigma[\,g_\infty\,], \tag{5.7}$$

$$\Sigma[\,g_\infty\,]=\int\limits_{|x|\to\infty}d\sigma^\mu\,\delta^{\alpha\beta}\Big(\partial_\alpha g_{\beta\mu}-\partial_\mu g_{\alpha\beta}\Big). \tag{5.8}$$

[6]This hope is based on a simple fact that the leading order of the $1/s$-expansion is not sensitive to the properties of the world function at all. Beyond this order the main object of interest, $\mathrm{Tr}\,K(s)$, involves the coincidence limit of the world function $\sigma(x,x)=0$, while its asymptotic coefficients in $\Omega_n(x,y)$ nonlocally depend on global geometry and can acquire from caustics additional contributions analogous to those of multiple geodesics connecting the points x and y beyond the geodesically convex neighborhood [21].

The proof of this statement cannot rely on the covariant expansion in powers of V and $R_{\mu\nu}$ [9, 11, 12, 13, 18]. This is because, in contrast to the perturbation theory in potential, the n-th term of this curvature expansion like in (3.18) is not explicitly available (even the calculation of the third order presents enormous calculational problem [12]). Therefore, a metric analogue of the variational equation (3.7) should be used to recover the functional trace from the kernel $K(s|\,x,y)$. But even this does not turn out to be sufficient, because the variational equation (3.7) is valid, strictly speaking, only up to surface terms, which do not vanish in the case of the metric variation. Indeed, the variational equation for the operator exponent actually reads

$$\delta_g \mathrm{Tr}\, e^{sF} = \int_0^s dt\, \mathrm{Tr}\, \left[e^{(s-t)F}\delta_g F e^{tF}\right] = s\mathrm{Tr}\, \left[\delta_g F e^{sF}\right] + \text{surface terms}, \tag{5.9}$$

where the second equality holds due to the cyclic property of the functional trace. Unlike for finite-dimensional matrices, here the cyclic property of the infinite-dimensional functional trace is based on multiple (actually infinitely multiple) integration by parts and, therefore, can bring nonvanishing surface integrals. Fortunately, for asymptotically-flat spacetime they can be completely taken into account within the first order of perturbation theory. Thus, for the derivation of (5.7) we will use the combination of the variational and perturbation techniques.

Consider first the metric variational derivative of $\mathrm{Tr}\, K(s)$ in the class of variations $\delta g_{\mu\nu}$ sufficiently rapidly decaying at spacetime infinity, so that no surface terms are arising while integrating by parts in the expression above. The corresponding variational derivative then takes the form

$$\begin{aligned}\frac{\delta \mathrm{Tr}\, K(s)}{\delta g_{\mu\nu}(x)} &= s\int dy\, \frac{\delta F(\nabla_y)}{\delta g_{\mu\nu}(x)}\, K(s|y,y')\Big|_{y'=y}\\ &= -s\, g^{1/2}(x)\, f^{\mu\nu}(\nabla_x,\nabla_y)K(s|x,y)\,\Big|_{x=y},\end{aligned} \tag{5.10}$$

where

$$f^{\mu\nu}(\nabla_x,\nabla_y) = -\nabla_x^{(\mu}\nabla_y^{\nu)} + \frac{1}{2}\, g^{\mu\nu}\Box_x + \frac{1}{2}\, g^{\mu\nu}\nabla_x^\lambda\nabla^y_\lambda. \tag{5.11}$$

Direct check then shows that in this class of variations which probe only the bulk part of (5.7) Eq.(5.10) is satisfied in the leading order of the $1/s$-expansion [15]

$$\frac{\delta W_0}{\delta g_{\mu\nu}(x)} = -g^{1/2}(x)\, f^{\mu\nu}(\nabla_x,\nabla_y)\,\Phi(x)\Phi(y)\,\Big|_{x=y}. \tag{5.12}$$

The variation of W_0 given by Eq.(5.7) is based here on the metric variational derivative of $\Phi(x)$ – the analogue of Eq.(3.11)

$$\frac{\delta\Phi(x)}{\delta g_{\mu\nu}(y)} = G(x,y)\, f^{\mu\nu}(\overrightarrow{\nabla}_y,\overleftarrow{\nabla}_y)\,\Phi(y), \tag{5.13}$$

where the arrow indicates in which direction the corresponding derivative is acting.

Metric variations decaying at $|x|\to\infty$ as asymptotically-flat corrections (5.4), $\delta g_{\mu\nu}(x)\sim 1/|x|^{d-2}$, induce nonvanishing surface terms in the variational equation (5.9)

and, therefore, they cannot be checked with the use of (5.10). Fortunately, the covariant perturbation theory of Sect.2.3 shows [11] that these terms (arising from cyclic permutations of heat kernels under the sign of the functional trace) appear only in the linear order of perturbation expansion in $h_{\mu\nu}$ – deviation of the metric from the flat-space one. Therefore, they can be recovered by simply comparing (5.7) with the late time asymptotics obtained in [12, 13] to cubic order in curvature and potential[7]. This asymptotics for the operator of the form (1.5)-(1.6) reads

$$\begin{aligned}
\mathrm{Tr}\,K(s) = &-\frac{s}{(4\pi s)^{d/2}}\int dx\, g^{1/2}\,\Big\{V + V\frac{1}{\Box}V + V\frac{1}{\Box}V\frac{1}{\Box}V + O\left(V^4\right)\Big\}\\
&+\frac{1}{6}\frac{s}{(4\pi s)^{d/2}}\int dx\, g^{1/2}\,\Big\{R - R_{\mu\nu}\frac{1}{\Box}R^{\mu\nu} + \frac{1}{2}R\frac{1}{\Box}R\\
&\qquad +\frac{1}{2}R\left(\frac{1}{\Box}R^{\mu\nu}\right)\frac{1}{\Box}R_{\mu\nu} - R^{\mu\nu}\left(\frac{1}{\Box}R_{\mu\nu}\right)\frac{1}{\Box}R\\
&\qquad +\left(\frac{1}{\Box}R^{\alpha\beta}\right)\left(\nabla_\alpha\frac{1}{\Box}R\right)\nabla_\beta\frac{1}{\Box}R\\
&\qquad -2\left(\nabla^\mu\frac{1}{\Box}R^{\nu\alpha}\right)\left(\nabla_\nu\frac{1}{\Box}R_{\mu\alpha}\right)\frac{1}{\Box}R\\
&\qquad -2\left(\frac{1}{\Box}R^{\mu\nu}\right)\left(\nabla_\mu\frac{1}{\Box}R^{\alpha\beta}\right)\nabla_\nu\frac{1}{\Box}R_{\alpha\beta} + \mathrm{O}\,[\,R^4_{\mu\nu}\,]\Big\}.
\end{aligned} \tag{5.14}$$

The nonlocal expansion of $\Phi(x)$

$$\Phi(x) = 1 + \frac{1}{\Box}V(x) + \frac{1}{\Box}V\frac{1}{\Box}V(x) + O\left(V^3\right) \tag{5.15}$$

obviously recovers from the first integral here the first term of (5.7) explicitly containing only powers of potential with *metric-dependent* nonlocalities.

The second integral in (5.14) is a topological invariant independent of local metric variations in the interior of spacetime – exactly in this class of $\delta g_{\mu\nu}(x)$ the functional derivative of (5.10) was calculated above. Direct expansion in powers of $h_{\mu\nu}$, $g_{\mu\nu} = \delta_{\mu\nu}+h_{\mu\nu}$, on flat-space background in cartesian coordinates shows that this term reduces to the surface integral at spacetime infinity. For the class of asymptotically flat metrics with $h_{\mu\nu}(x) \sim 1/|x|^{d-2}$, $|x| \to \infty$, this surface integral is linear in perturbations (contributions of higher powers of $h_{\mu\nu}$ to this integral vanish) and involves only a *local*

[7] This approximation is more than sufficient to check the term linear in metric perturbation $h_{\mu\nu}$. In covariant perturbation theory this perturbation gets expanded in the infinite power series in curvatures – this is the price one pays for having this expansion generally covariant.

asymptotic behavior of the metric $g^{\infty}_{\mu\nu}(x) = \delta_{\mu\nu} + h_{\mu\nu}(x)\,\Big|_{|x|\to\infty}$,

$$\int dx\, g^{1/2} \Big\{ R - R_{\mu\nu}\frac{1}{\Box}R^{\mu\nu} + \frac{1}{2}\,R\frac{1}{\Box}R$$
$$+\frac{1}{2}\,R\left(\frac{1}{\Box}R^{\mu\nu}\right)\frac{1}{\Box}R_{\mu\nu} - R^{\mu\nu}\left(\frac{1}{\Box}R_{\mu\nu}\right)\frac{1}{\Box}R$$
$$+\left(\frac{1}{\Box}R^{\alpha\beta}\right)\left(\nabla_\alpha\frac{1}{\Box}R\right)\nabla_\beta\frac{1}{\Box}R$$
$$-2\left(\nabla^\mu\frac{1}{\Box}R^{\nu\alpha}\right)\left(\nabla_\nu\frac{1}{\Box}R_{\mu\alpha}\right)\frac{1}{\Box}R$$
$$-\,2\left(\frac{1}{\Box}R^{\mu\nu}\right)\left(\nabla_\mu\frac{1}{\Box}R^{\alpha\beta}\right)\nabla_\nu\frac{1}{\Box}R_{\alpha\beta} + \mathrm{O}[\,R^4_{\mu\nu}\,]\Big\}$$
$$= \int\limits_{|x|\to\infty} d\sigma^\mu\,\Big(\partial^\nu h_{\mu\nu} - \partial_\mu h\Big) \equiv \Sigma\,[\,g_\infty\,]. \qquad (5.16)$$

Here $d\sigma^\mu$ is the surface element on the sphere of radius $|x| \to \infty$, $\partial^\mu = \delta^{\mu\nu}\partial_\nu$ and $h = \delta^{\mu\nu}h_{\mu\nu}$. Covariant way to check this relation is to calculate the metric variation of this integral and show that its integrand is the total divergence which yields the surface term of the above type linear in $\delta g_{\mu\nu}(x) = h_{\mu\nu}(x)$. This is explicitly done in Appendix B. Thus, the correct expression for W_0 is indeed modified by the the surface integral $\Sigma\,[\,g_\infty\,]$, and this integral does not contribute to the metric variational derivative $\delta W_0/\delta g_{\mu\nu}(x)$ at finite $|x|$.

For asymptotically-flat metrics with a power-law falloff at infinity $h_{\mu\nu}(x) \sim M/|x|^{d-2}$, $|x| \to \infty$, the surface integral $\Sigma\,[\,g_\infty]$ forms the contribution to the Einstein action

$$S_{\rm E}[\,g\,] \equiv -\int dx\, g^{1/2}\, R(g) + \Sigma\,[\,g_\infty], \qquad (5.17)$$

which guarantees the correctness of the variational procedure leading to Einstein equations. Covariantly this integral can also be rewritten in the Gibbons-Hawking form $S_{GH}\,[\,g\,] = \Sigma\,[\,g_\infty]$ – the double of the extrinsic curvature trace K on the boundary (with a properly subtracted infinite contribution of the flat-space background) [22]

$$\Sigma\,[\,g_\infty] = -2\int_\infty d^{d-1}\sigma\,\Big(g^{(d-1)}\Big)^{1/2}\Big(K - K_0\Big). \qquad (5.18)$$

Thus, this is the surface integral of the *local* function of the boundary metric and its normal derivative. The virtue of the relation (5.16) is that it expresses this surface integral in the form of the spacetime (bulk) integral of the *nonlocal* functional of the bulk metric. The latter does not explicitly contain auxiliary structures like the vector field normal to the boundary, though these structures are implicitly encoded in boundary conditions for nonlocal operations in the bulk integrand of (5.16).

Note also, in passing, that the relation (5.16) can be used to rewrite the (Euclidean) Einstein-Hilbert action (5.17) as the *nonlocal* curvature expansion which begins with the *quadratic* order in curvature. This observation serves as a basis for covariantly consistent nonlocal modifications of Einstein theory [10] motivated by the cosmological constant and cosmological acceleration problems [23].

5.2. Conformal properties

Important case of the *metric-dependent* potential in (1.5)-(1.6) corresponds to the operator of the conformal scalar field. In d dimensions it reads

$$F(\nabla) = \Box - \frac{1}{4}\,\frac{d-2}{d-1}\,R. \tag{5.19}$$

Under local conformal (Weyl) transformations of the metric tending to identity at infinity

$$\begin{aligned}
&\bar{g}_{\mu\nu}(x) = \Psi^{4/(d-2)}(x)\, g_{\mu\nu}(x),\\
&\Psi(x) = 1 + O\left(\frac{1}{|x|^{d-2}}\right), \quad |x| \to \infty
\end{aligned} \tag{5.20}$$

this operator transforms homogeneously by multiplication from the left and from the right with certain powers of the conformal factor

$$\bar{F}(\nabla) = \Psi^{-1-4/(d-2)}(x)\, F(\nabla)\, \Psi(x). \tag{5.21}$$

In virtue of this property, the universal function $\Phi(x)$, (5.5), also transforms homogeneously, because it solves the homogeneous equation with the operator (5.19) and satisfies the same unit boundary conditions,

$$\bar{\Phi}(x) \equiv \Phi(x)[\,\bar{g}_{\mu\nu}] = \frac{\Phi(x)}{\Psi(x)}. \tag{5.22}$$

One more interesting property is the conformal transformation of the scalar curvature which reads

$$\bar{R}(x) = -4\,\frac{d-1}{d-2}\,\Psi^{-1-4/(d-2)}(x)\,\Big[\,F(\nabla)\Psi(x)\,\Big] \tag{5.23}$$

and, thus, means that $\Phi(x)$ can be regarded as a special case of the conformal transformation, $\Psi = \Phi$, to the conformal gauge of vanishing scalar curvature, $\bar{R} = 0$.

Consider now the bulk part of the leading asymptotics W_0, (5.7), for the conformal operator (5.19)

$$W_0[\,g\,] = -\frac{1}{4}\,\frac{d-2}{d-1}\int dx\, g^{1/2} R\,\Phi + \frac{1}{6}\,\Sigma[\,g_\infty]. \tag{5.24}$$

In virtue of (5.23) it transforms as

$$\begin{aligned}
-\frac{1}{4}\,\frac{d-2}{d-1}\int dx\,\bar{g}^{1/2}\bar{R}\,\bar{\Phi} &= \int dx\, g^{1/2}\Phi\, F(\nabla)\Psi\\
&= \int_\infty d\sigma^\mu\,(\partial_\mu\Psi - \partial_\mu\Phi).
\end{aligned} \tag{5.25}$$

Here we integrated by parts, used the equation $F(\nabla)\Phi = 0$ and took into account that Φ and Ψ equal one at infinity (but their derivatives generically do not tend to zero fast

enough to discard the surface integral). The second term of the surface integral can be transformed back to the form of the bulk integral (not involving Ψ now)

$$-\int_\infty d\sigma^\mu\,\partial_\mu\Phi = -\int dx\, g^{1/2}\,\Box\Phi = -\frac{1}{4}\,\frac{d-2}{d-1}\int dx\, g^{1/2} R\,\Phi, \tag{5.26}$$

whence finally

$$\int dx\,\bar g^{1/2}\bar R\,\bar\Phi = \int dx\, g^{1/2} R\,\Phi - 4\,\frac{d-1}{d-2}\int_\infty d\sigma^\mu\,\partial_\mu\Psi. \tag{5.27}$$

Together with the conformal transformation of the Gibbons-Hawking integral (5.8),

$$\bar\Sigma \equiv \Sigma[\,\bar g_\infty] = \Sigma - 4\,\frac{d-1}{d-2}\int_\infty d\sigma^\mu\,\partial_\mu\Psi, \tag{5.28}$$

this means that the following linear combination

$$S_{\rm C}[\,g\,] = -\int dx\, g^{1/2} R\,\Phi + \Sigma[\,g_\infty] \tag{5.29}$$

is conformally invariant in the class of local Weyl transformations (5.20)

$$S_{\rm C}[\,g\,] = S_{\rm C}[\,\bar g\,]. \tag{5.30}$$

This particular combination can be obtained from the Einstein action (5.17) by transition to the conformal gauge of vanishing scalar curvature. By performing the transformation (5.20) with $\Psi = \Phi[\,g\,]$ one finds that

$$S_{\rm C}[\,g_{\mu\nu}] = S_{\rm E}\left[\,g_{\mu\nu}\Phi^{4/(d-2)}[\,g\,]\,\right]. \tag{5.31}$$

Conformal invariance of this expression is obvious because $g_{\mu\nu}\Phi^{4/(d-2)}[\,g\,]$ is the invariant of the conformal transformation (5.20). In fact, this functional in 4-dimensional context was suggested in [24] as a conformal off-shell extension of the Einstein theory in vacuum (or with traceless matter sources).

Consider now the conformal properties of the heat kernel asymptotics for the operator (5.19). For the heat kernel itself they directly follow from the transformation of the function $\Phi(x)$ (5.22) and read in the leading order as

$$\bar\Omega(x,y)\,\bar g^{1/2}(y) = \Psi^{-1}(x)\,\Big[\,\Omega(x,y)\,g^{1/2}(y)\,\Big]\,\Psi^{1+4/(d-2)}(y). \tag{5.32}$$

This transformation looks similar to that of the operator (5.21). However, this coincidence is accidental, because unlike (5.21) the heat kernel does not transform by a simple homogeneous law. Equation (5.21) does not represent the *similarity* transformation – the factors on the left and right of $F(\nabla)$ in (5.21) are not inverse to one another. The extra factor $\Psi^{-4/(d-2)}$ in $\bar F(\nabla) = \Psi^{-1}\left(\Psi^{-4/(d-2)}\,F(\nabla)\right)\Psi$ indicates that the operator $\bar F(\nabla)$ is similar to the other operator $\Psi^{-4/(d-2)}\,F(\nabla)$ having another heat kernel. In fact, this factor, which determines the conformal rescaling of metric (5.20), leads to the exactly calculable conformal anomaly of the effective action. But for the heat kernel itself and its functional trace it generates a nontrivial transformation law. For the

functional trace this transformation with the infinitesimal $\delta\Psi(x)$ in $\Psi(x) = 1 + \delta\Psi(x)$ is

$$\delta_\Psi \mathrm{Tr}\, \bar{K}(s)\Big|_{\Psi=1} = -\frac{4}{d-2}\, s\frac{d}{ds}\, \mathrm{Tr}\,\Big(\delta\Psi\, K(s)\Big)$$
$$= -\frac{4}{d-2}\, s\frac{d}{ds} \int dx\, \delta\Psi(x)\, \Omega(s\,|\,x,x). \tag{5.33}$$

When $\delta\Psi(x)$ has a compact support we can use here the leading order asymptotics $\Omega(s\,|\,x,x) \sim \Phi^2(x)/(4\pi s)^{d/2}$, because in this case the domain of $|x| \to \infty$ (which breaks the uniformity of this asymptotics) does not enter the range of integration, and $\delta_\Psi \mathrm{Tr}\, K(s) = O(1/s^{d/2})$ at $s \to \infty$. Therefore, the leading asymptotics W_0 is conformal invariant for Weyl rescalings with a compact support. Indeed, both of the terms in (5.37) are separately invariant in view of (5.27) and (5.28).

For generic conformal transformations with $\delta\Psi(x) \to 0$ but $\delta\Psi(x) \neq 0$ at $|x| \to \infty$ this conclusion cannot be inferred on the basis of (5.33) because it starts involving the domain where the non-uniform asymptotics of $\Omega(s\,|\,x,x)$ breaks down. Therefore, for such transformations W_0 should not necessarily be conformal invariant, which explains the mismatch of coefficients of the bulk and surface terms in (5.37) as compared to the conformal extension of the Einstein action (5.29).

Quite interestingly, in the exceptional case of the 4-dimensional spacetime, this invariance holds for W_0 even for transformations with non-compact support. In this case, the coefficient of nonminimal curvature-scalar coupling $(d–2)/4(d–1) = 1/6$ and

$$W_0[\,g\,] = \frac{1}{6}\, S_{\rm C}[\,g\,], \quad d = 4. \tag{5.34}$$

5.3. Problems with the subleading order

Unfortunately, inclusion of gravity results in the number of problems in the subleading order of late time expansion. The equation for $\Omega_1(x,y)$ generalizing (3.13) takes the form

$$F(\nabla)\, \Omega_1(x,y) = \left(\sigma^\mu(x,y)\nabla_\mu + \frac{1}{2}\,\Box\,\sigma(x,y) - \frac{d}{2}\right)\Phi(x)\,\Phi(y) \tag{5.35}$$

and has a symmetric solution similar to (3.14) [15]

$$\Omega_1(x,y) = \frac{1}{2}\,\psi(x,y) + \frac{1}{2}\,\psi(y,x)$$
$$-\frac{1}{2}\,\frac{1}{F(\nabla_x)}\,\overrightarrow{F}(\nabla_x)\,[\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,]\,\overleftarrow{F}(\nabla_y)\overleftarrow{\frac{1}{F(\nabla_y)}}. \tag{5.36}$$

Here $\psi(x,y)$ is a special two-point function

$$\psi(x,y) = \frac{1}{F(\nabla_x)}\Big[\,2\sigma^\mu(x,y)\nabla_\mu\Phi(x) + (\Box\,\sigma(x,y) - d)\,\Phi(x)\Big]\,\Phi(y) \tag{5.37}$$

and the arrows indicate the action of the differential operators in the direction opposite to Green's functions, $1/F(\nabla)$, written in the operator form. This certainly implies that

the integration by parts that would reverse the action of $F(\nabla)$ on $1/F(\nabla)$ (and, thus, would lead to a complete cancellation of the corresponding nonlocality) is impossible without generating nontrivial surface terms.

As shown in [15], with this expression for the $\Omega_1(x,y)$ the variational equation for W_1,

$$\frac{\delta W_1}{\delta V(x)} = -g^{1/2}(x)\,\Omega_1(x,x), \tag{5.38}$$

has the following formal solution in terms of the Green's function of $F(\nabla)$

$$W_1[\,V,g_{\mu\nu}] = \frac{1}{2}\int dx\, g^{1/2}(x)\,\frac{1}{F(\nabla_x)}\,\overrightarrow{F}(\nabla_x)\,\Big[\Phi(x)\,\sigma(x,y)\,\Phi(y)\Big]\,\overleftarrow{F}(\nabla_y)\,\Big|_{\,y=x} \\ +W_1^{\rm metric}[\,g_{\mu\nu}]. \tag{5.39}$$

Here $W_1^{\rm metric}[\,g_{\mu\nu}]$ is some purely metric functional – a functional integration "constant" for equation (5.38). The latter can be determined only from the metric variational equation (5.10). Quite interestingly, this equation confirms the above expression with a purely constant metric-independent $W_1^{\rm metric}[\,g_{\mu\nu}] = {\rm const}$ (see Appendix D).

Unfortunately, the validity of the algorithms (5.36), (5.37) and (5.39) can be rigorously established only in flat spacetime. Problem is that the nonlocal function $\psi(x,y)$ is well (and uniquely) defined only when the expression in square brackets of (5.37) sufficiently rapidly goes to zero at spacetime infinity. This expression has two terms, the first of which has a power law falloff $1/|x|^{d-2}$ at $|x|\to\infty$ in view of the behavior of $\sigma^\mu(x,y)\sim|x|$ and $\nabla_\mu\Phi(x)\sim 1/|x|^{d-1}$. This makes the contribution of this term (convolution with the kernel of Green's function) well defined at least in dimensions $d>4$. On the contrary, the second term is proportional to the deviation of geodesics $\Box\,\sigma(x,y)-d$ which has the following rather moderate falloff

$$\Box\,\sigma(x,y)-d\sim\frac{1}{|x|},\quad |x|\to\infty. \tag{5.40}$$

Therefore a purely metric contribution to (2.12) turns out to be quadratically divergent in the infrared

$$\frac{1}{F(\nabla_x)}\,(\Box\,\sigma(x,y)-d)\Phi(x)\,\Phi(y) \\ =\int dx'\,G(x,x')(\Box\,\sigma(x',y)-d)\Phi(x')\,\Phi(y)=\infty. \tag{5.41}$$

Tracing the origin of this difficulty back to the equation (5.35) we see that the source term in its right hand side is $O(1/|x|)$, so that the solution $\Omega_1(x,y)\sim|x|$ is not vanishing at infinity and, therefore, is not uniquely fixed by Dirichlet boundary conditions. Some principles of fixing this ambiguity would certainly regularize the integral in the definition of $\psi(x,y)$ and uniquely specify all quantities in the subleading order, but this remains the problem for future.

In flat spacetime the geodesic deviation scalar (5.40) is identically vanishing, because $\sigma(x,y)=|x-y|^2/2$, $\sigma^\mu(x,y)=(x-y)^\mu$, $\Box\,\sigma(x,y)=d$. Therefore, the expression

for $\psi(x,y)$ becomes well defined. Correspondingly, in the square brackets of (5.39) only one term containing $\nabla^\mu_x\nabla^\nu_y\sigma(x,y)=-\delta^{\mu\nu}$ survives and yields

$$F(\nabla_x)F(\nabla_y)\Phi(x)\,\sigma(x,y)\,\Phi(y)=-4\,\nabla_\mu\Phi(x)\,\nabla^\mu\Phi(y), \tag{5.42}$$

so that $\Omega_1(x,y)$ and W_1 reduce to the expressions (3.14) and the second term of (3.15) correspondingly.

There is a conspicuous mismatch between (3.15) and the flat-space limit of (5.39) – the first unit term of (3.15). This term is missing in (5.39) and the attempt to identify it with the constant,

$$W_1^{\rm metric}=\int dx\,\times 1, \tag{5.43}$$

contradicts the covariantization procedure in the transition to curved spacetime (actually confirmed within the covariant curvature expansion of [11, 12])

$$\mathrm{Tr}\,K(s)=\frac{1}{(4\pi s)^{d/2}}\int dx\,\times 1+...\to\frac{1}{(4\pi s)^{d/2}}\int dx\,g^{1/2}(x)\times 1+...\,. \tag{5.44}$$

The metric-independent integral (5.43) seems counterintuitive because of its noncovariance. The origin of this situation may be ascribed to the difficulties of the above type. The derivation of (5.39) is based on the operation with unregulated divergent integrals which casts serious doubt on its validity. However, the fact that it formally passes a subtle check of the metric variation (see Appendix D) suggests that under certain regularization of these infrared divergences (5.39), (5.43) the algorithm will survive beyond flat spacetime. The compatibility of this algorithm with the covariant perturbation expansion of [11, 12] might be nontrivial in view of these divergences and based on disentangling the covariant integral $\int dx\,g^{1/2}\times 1$ from divergent nonlocal structures of (5.39), (5.43). This procedure is discussed in the concluding section below.

The infrared divergence of the integral $\int dx\,g^{1/2}\times 1$ reflects the continuity of the spectrum of the operator $F(\nabla)$, and in flat spacetime represents a trivial constant which does not affect physical predictions. In curved spacetime this integral becomes a functional of geometry and, thus, incorporates the cosmological constant term. Therefore, the subleading order of the late-time expansion gets intertwined with the cosmological constant problem, which we briefly discuss in Conclusions.

6. Discussion and conclusions

Main results presented above include the nonperturbative heat kernel asymptotics (3.10) and (3.12), their generalization to curved spacetime (5.7)-(5.8), and as a byproduct new nonlocal effective action algorithms (4.8), (4.10) and (4.13). Let us now briefly discuss potential generalizations and applications of these results and the remaining loopholes in our technique.

Nonperturbative effective action alternative to the Coleman-Weinberg potential is very interesting in various applications. Already at this preliminary stage it is clear

that the algorithms (4.8) and (4.10) show nontrivial and qualitatively important dependence on spacetime dimensionality, because in $d > 4$ the action is dominated by the *renormalization-independent* structure not involving the parameter μ^2. This is very different from the four-dimensional case (4.8) when the infrared effects simply "delocalize" the Coleman-Weinberg potential, but leave the dominant dependence on the ultraviolet scale parameter. This means that scaling arguments cannot grasp this infrared dominant term at all. It is also interesting that this term is *negative-definite*, which might have important implications. At the same time, the nonperturbative terms deserve further analysis ascertaining their range of applicability, the corrections due to local derivative factors completely disregarded above (terms with $\tilde{a}_n(x,x)$ for $n > 0$).

Regarding the general formalism of late time expansion it should be emphasized that the difficulty we encountered in the subleading order indicates a serious problem. This difficulty will proliferate in higher orders of the $1/s$-expansion, for which the coefficients $\Omega_n(x,y)$ at spacetime infinity will behave as higher and higher powers in $|x|$ and $|y|$ and thus comprise increasingly more complicated boundary value problems. This will generally happen not only in metric sector, but in flat spacetime as well.

The subleading order is free from this problem in flat spacetime, because the expressions (3.14) and (3.15) are well defined, but inclusion of gravity results in the problem discussed above. However, already this order of $1/s$-expansion is very interesting, because it incorporates the cosmological constant problem. Indeed, the cosmological term is generated via the integral (1.10) from the (covariantized) unit term in $\mathrm{Tr}\, K(s)$, (5.44), as

$$\mathit{\Gamma}_\Lambda = \int dx\, g^{1/2}\, \Lambda_\infty, \quad \Lambda_\infty = -\frac{1}{2(4\pi)^{d/2}} \int_0^\infty \frac{ds}{s^{1+d/2}} \times 1. \tag{6.1}$$

The cosmological constant Λ_∞ here is ultraviolet divergent, and this expression is also infrared divergent in the coordinate sense[8] – the volume integral $\int dx\, g^{1/2}$ for asymptotically-flat spacetime diverges at $|x| \to \infty$.

Therefore, the covariance problem for W_1 with metric-independent W_1^{metric}, given by Eqs. (5.39) and (5.43), amounts to correctly recovering the covariant cosmological term from the nonlocal (and divergent) expressions. To make the discussion of this point simpler, consider a purely metric case of vanishing potential, $V = 0$, $\Phi = 1$. For this case the subleading term of the functional trace (with all the reservations discussed in Sect.5.3) is anticipated to be

$$W_1 = \int dx \times 1 + \frac{1}{2} \int dx\, dy\, g^{1/2}(x)\, G(x,y)\, \Box_x \Box_y \sigma(x,y). \tag{6.2}$$

The second integral obviously vanishes in flat spacetime where $\Box_x \Box_y \sigma(x,y) = 0$, so that its curvature expansion should begin with the first order. This integral is infrared

[8] Of course, the variety of divergences indicates that the cosmological constant cannot consistently arise in asymptotically-flat spacetime. The contribution (6.1) in *massless* theories does not carry any sensible physical information and is cancelled due to a number of interrelated mechanisms. Its cancellation is guaranteed by the contribution of the local path-integral measure to the effective action, which annihilates strongest (volume) divergences [25]. Another mechanism is based on the use of the dimensional regularization which puts to zero all power-like divergences. All these mechanisms, however, stop working for *massive* theories or for theories with spontaneously broken symmetry, where the induced vacuum energy presents a real hierarchy problem [26].

divergent and, moreover, as it follows from calculations of Appendix D, it is quadratically divergent, just like the first order term in the metric (or curvature) expansion of

$$\int dx\, g^{1/2}(x) = \int dx \left(1 + \frac{1}{2}\, h + ...\right) = \int dx \left(1 - \frac{1}{\Box} R(x) + ...\right). \qquad (6.3)$$

Comparison with (6.2) suggests that the second integral of (6.2) begins with this nonlocal term linear in $(1/\Box)R$ (or local in h). Thus the total cosmological term seems to be camouflaged in (6.2) as a sum of a trivial local and a special nonlocal functionals[9],

$$W_1 = \int dx\, g^{1/2} \times 1 + O\Big(\, R^2 \,\Big). \qquad (6.4)$$

This situation is not entirely new. We have already seen that the local surface integral of Gibbons and Hawking has a nonlocal representation (5.16) which, in its turn, underlies the nonlocal representation of the Einstein action of [10]. This analogy is, however, marred by the fact that the corresponding integrals in (6.2) and (6.3) are infrared divergent, and special regularization is needed to make their rigorous comparison. This regularization will apparently be a part of the scheme necessary for rendering the subleading order of the $1/s$-asymptotics (and maybe all of its higher orders) well-defined. The invention of this regularization will result in two possible outcomes – it will either confirm the equation (6.4) or show that the cosmological term $\int dx\, g^{1/2}$ enters $\mathrm{Tr}\, K(s)$ at $s \to 0$ and $s \to \infty$ with different coefficient functions of s.

The second option seems rather unlikely, but if it happens, this would mean a new mechanism of induced cosmological constant. Indeed, in this case the s-dependent (or partial) "cosmological constant" $\lambda(s)$ in $\mathrm{Tr}\, K(s)$, $\mathrm{Tr}\, K(s) = \lambda(s) \int dx\, g^{1/2} + ...$, would interpolate between $\lambda_<(s) = 1/(4\pi s)^{d/2}$ at $s \to 0$ and some other function $\lambda_>(s)$ at $s \to \infty$. This would result in the full induced cosmological constant

$$\Lambda_{\mathrm{ind}} = -\frac{1}{2} \int_0^\infty \frac{ds}{s}\, \lambda(s) \times 1 \neq 0, \qquad (6.5)$$

that will definitely be non-vanishing, because in contrast to (6.1) this integral no longer represents a pure power divergence annihilated by the dimensional regularization[10]. This possibility is currently under study. We expect that this might bring to light interesting interplay between the cosmological constant problem and infrared asymptotics of the heat kernel and nonlocal effective action.

Let us conclude by listing possible generalizations of our late time technique. Once the problems with subleading order are resolved, one can extend the effective action algorithms (4.8) and (4.10) to include curvature. This extension is interesting, because it might provide us with long-distance modifications of gravity theory characterized by the nonlocal scale-dependent gravitational "constant" [10, 23]. Another generalization consists in overstepping the limits of the asymptotically-flat spacetime. The first

[9] The idea of nonlocal representation of the cosmological term was also considered in [10, 27].

[10] This interesting mechanism will require additional dimensionful parameter, and maybe the absence of such parameter will preclude this mechanism from realization.

thing to do here is to consider asymptotically deSitter boundary conditions which are strongly motivated by the cosmological acceleration phenomenon and by the dS/CFT-correspondence conjecture [28]. This generalization implies essential modification of both perturbative and nonperturbative techniques for the heat kernel, the generalization of the Gibbons-Hawking term to asymptotically dS-spacetimes, etc. Another generalization concerns the inclusion of higher spins with the covariant derivatives in the d'Alembertian involving not only the metric connection but the gauge field connection as well. All these issues are currently under study and will be presented elsewhere.

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A. Late time asymptotics in perturbation theory

Substituting the expression (3.22) for Ω_n in (3.18) and expanding in powers of the term bilinear in α-parameters one gets

$$\mathrm{Tr}\, K_n(s) = \frac{(-s)^n}{(4\pi s)^{d/2}} \int dx \int_0^\infty d^{n-1}\alpha \, \exp\Big(s \sum_{i=2}^n \alpha_i D_i^2\Big) \times \left(1 - s \sum_{m,k=2}^n \alpha_m \alpha_k D_m D_k + ...\right) V_1 V_2 ... V_n. \tag{A.1}$$

Here $1/n$ factor disappeared due to the contribution of n equal terms corresponding to the operatorial maxima of Ω_n. Also the range of integration over $\alpha_2, ...\alpha_n$, $\sum_{i=2}^n \alpha_i \leq 1$, was extended to all positive values of α_i. This is justified since the error we make by this extension goes to higher orders of $1/s$-expansion[11]. The second term in the round brackets can be rewritten in terms of the derivatives with respect to D_m^2 acting on the exponential, so that

$$\mathrm{Tr}\, K_n(s) = \frac{(-s)^n}{(4\pi s)^{d/2}} \int dx \left(1 - \frac{1}{s} \sum_{m,k=2}^n D_m D_k \frac{\partial}{\partial D_m^2} \frac{\partial}{\partial D_k^2} + ...\right) \times \int_0^\infty d^{n-1}\alpha \, \exp\Big(s \sum_{i=2}^n \alpha_i D_i^2\Big) V_1 V_2 ... V_n. \tag{A.2}$$

[11] If Ω_n were not a differential operator this error would be exponentially small in $s \to \infty$. Because of the heat-kernel operator nature, however, it turns out to be suppressed by extra power-like factor $O(1/s^{d/2})$ and, therefore, goes beyond the leading and subleading orders.

In this form it is obvious that further terms of expansion in powers of the quadratic in α part of Ω_n bring higher order corrections of the $1/s$-series. Doing the integral over α here and performing differentiations one obtains

$$\begin{aligned}\mathrm{Tr}\,K_n(s) = \frac{1}{(4\pi s)^{d/2}}\int dx\,\Bigg[&-s\frac{1}{D_2^2...D_n^2} + 2\sum_{m=2}^{n}\frac{1}{D_2^2...D_{m-1}^2}\,\frac{1}{(D_m^2)^2}\,\frac{1}{D_{m+1}^2...D_n^2}\\ &+2\sum_{m=2}^{n-1}\sum_{k=m+1}^{n}\frac{1}{D_2^2...D_{m-1}^2}\,\frac{D_m^\mu}{(D_m^2)^2}\,\frac{1}{D_{m+1}^2...D_{k-1}^2}\,\frac{D_{k\mu}}{(D_k^2)^2}\,\frac{1}{D_{k+1}^2...D_n^2}\\ &+O\left(\frac{1}{s}\right)\Bigg]\,V_1V_2...V_n. \end{aligned}\tag{A.3}$$

The first term in the square brackets gives the leading order term of the late time expansion. It can be further transformed by taking into account that any operator D_m defined by (3.23) acts as a partial derivative only on the group of factors $V_mV_{m+1}...V_n$ in the full product $V_1...V_n$, $D_mV_1...V_n = V_1...V_{m-1}\nabla(V_{m+1}...V_n)$. Therefore all the operators understood as *acting to the right* can be ordered in such a way

$$\mathrm{Tr}\,K_n(s) = -\frac{s}{(4\pi s)^{d/2}}\int dx\,V_1\frac{1}{D_n^2}V_n\frac{1}{D_{n-1}^2}V_{n-1}...\frac{1}{D_2^2}V_2 + O\left(\frac{1}{s^{d/2}}\right)\tag{A.4}$$

that the labels of D_m^2's can be omitted and all D_m^2 can be identified with boxes also acting to the right

$$\mathrm{Tr}\,K_n(s) = -\frac{s}{(4\pi s)^{d/2}}\int dx\,\underbrace{V\frac{1}{\Box}V\frac{1}{\Box}...V\frac{1}{\Box}}_{n-1}\,V(x) + O\left(\frac{1}{s^{d/2}}\right).\tag{A.5}$$

Infinite summation of this series is not difficult to perform because this is the geometric progression in powers of the nonlocal operator $V(1/\Box)$

$$\mathrm{Tr}\,K(s) = \mathrm{Tr}\,K_0(s) - \frac{s}{(4\pi s)^{d/2}}\int dx\,\sum_{n=0}^{\infty}\left(V\frac{1}{\Box}\right)^n V(x) + O\left(\frac{1}{s^{d/2}}\right),\tag{A.6}$$

which gives rise to the leading order in (3.21).

The subleading in s terms are given by infinite resummation over n of the second and third terms in square brackets of Eq.(A.3). Remarkably, this summation can again be explicitly done. In this case one has to sum multiple geometric progressions.

The second term of (A.3) gives rise to the series

$$\frac{2}{(4\pi s)^{d/2}}\int dx\,\sum_{n=2}^{\infty}\sum_{m=2}^{n}V\,\underbrace{\frac{1}{\Box}V...\frac{1}{\Box}V}_{n-m}\,\frac{1}{\Box^2}\,\underbrace{V\frac{1}{\Box}...V\frac{1}{\Box}}_{m-2}\,V(x).\tag{A.7}$$

By summing the two geometric progressions with respect to independent summation indices $0 \le n-m < \infty$ and $0 \le m-2 < \infty$ one finds that this series reduces to

$$\frac{2}{(4\pi s)^{d/2}}\int dx\,V\,\frac{1}{(\Box - V)^2}\,V(x),\tag{A.8}$$

which after the integration by parts amounts to

$$\frac{2}{(4\pi s)^{d/2}} \int dx \left(\frac{1}{\Box - V} V(x) \right)^2 = \frac{2}{(4\pi s)^{d/2}} \int dx \Big(1 - \Phi(x) \Big)^2. \tag{A.9}$$

Similarly, the third term of (A.3) gives rise to the triplicate geometric progression which after summation and integration by parts reduces to

$$\frac{2}{(4\pi s)^{d/2}} \int dx\, V \sum_{i=0}^{\infty} \Big(\frac{1}{\Box} V\Big)^i \frac{1}{\Box} \nabla^\mu \frac{1}{\Box} \sum_{j=0}^{\infty} \Big(V \frac{1}{\Box}\Big)^j \frac{1}{\Box} \nabla^\mu \frac{1}{\Box} \sum_{l=0}^{\infty} \Big(V \frac{1}{\Box}\Big)^l V(x)$$
$$= -\frac{2}{(4\pi s)^{d/2}} \int dx \Big(\nabla_\mu \Phi(x) \Big) \frac{1}{\Box - V} V \frac{1}{\Box} \nabla^\mu \Phi(x). \tag{A.10}$$

Taking here into account that

$$\frac{1}{\Box - V} V \frac{1}{\Box} = \frac{1}{\Box - V} - \frac{1}{\Box} \tag{A.11}$$

one finds that the sum of (A.9) and (A.10) is equal to

$$\frac{2}{(4\pi s)^{d/2}} \int dx \left((1 - \Phi)^2 - \nabla_\mu \Phi \frac{1}{\Box - V} \nabla^\mu \Phi + \nabla_\mu \Phi \frac{1}{\Box} \nabla^\mu \Phi \right)$$
$$= -\frac{2}{(4\pi s)^{d/2}} \int dx\, \nabla_\mu \Phi \frac{1}{\Box - V} \nabla^\mu \Phi, \tag{A.12}$$

where the cancellation of the first and the third terms takes place after rewriting $\nabla_\mu \Phi$ in the third term as $\nabla_\mu(\Phi - 1)$ and integrating it by parts[12]. This finally recovers the subleading term of (3.26).

B. Nonperturbative effective action

To generate effective action by piecewise smooth approximation (4.1)-(4.5) we first note that the equation (4.1) for s_*,

$$\int dx\, \exp\left(-V s_*\right) = \int dx\, (1 - s_* V \Phi), \tag{B.1}$$

is not exactly solvable. Its solution, as some nontrivial functional of the potential, $s_* = s_*[\,V(x)\,]$, can however be approximately obtained for two wide classes of potentials (4.6) satisfying the bounds (4.8) and (4.12) respectively. As we will now see these bounds also provide the efficiency of this approximation.

For this purpose we rewrite the action (4.5) as a sum of two contributions

$$\bar{\varGamma} = \varGamma_< + \varGamma_> = -\frac{1}{2} \int_0^\infty \frac{ds}{s} \mathrm{Tr}\, K_<(s) - \frac{1}{2} \int_{s_*}^\infty \frac{ds}{s} \left(\mathrm{Tr}\, K_>(s) - \mathrm{Tr}\, K_<(s) \right). \tag{B.2}$$

[12]Straightforward integration by parts of $\nabla_\mu \Phi (1/\Box) \nabla^\mu \Phi$ is impossible because $\Phi(x)$ does not vanish at $|x| \to \infty$, while $\Phi(x) - 1$ does.

The first integral here represents the calculation of Sect.2.2 within the modified Schwinger-DeWitt expansion with $\tilde{a}_0 = 1$ and $\tilde{a}_n = 0$, $n \geq 1$, which in our particular case gives rise to

$$\mathit{\Gamma}_< = \mathit{\Gamma}_{\rm CW}, \tag{B.3}$$

where $\mathit{\Gamma}_{\rm CW}$ is just the Coleman-Weinberg potential (2.15) and we disregard the divergent part and all finite terms reflecting renormalization ambiguity. The second integral in (B.2) can also be calculated exactly. Integrating by parts and taking into account (B.1), we obtain

$$\begin{aligned}\mathit{\Gamma}_> &\equiv -\frac{1}{2}\int\limits_{s_*}^{\infty}\frac{ds}{s}\,\big(\mathrm{Tr}\,K_>(s) - \mathrm{Tr}\,K_<(s)\big)\\ &= \frac{1}{2(4\pi s_*)^{d/2}}\int dx\,\Big[\,\frac{4\,s_* V\Phi}{d\,(d-2)} + \sum_{n=1}^{d/2-1}\frac{(d/2-n-1)!}{(d/2)!}\,(-s_*V)^n\,e^{-s_*V}\\ &\qquad\qquad + \frac{(-s_*V)^{d/2}}{(d/2)!}\Gamma(0,s_*V)\,\Big],\end{aligned} \tag{B.4}$$

where $\Gamma(0,x)$ is an incomplete gamma function, $\Gamma(0,x) = \int_x^\infty dt\,t^{-1}e^{-t}$, with the following asymptotics

$$\Gamma(0,x) \sim \begin{cases}\ln\dfrac{1}{x}\,, & x \ll 1,\\[2mm] \dfrac{1}{x}\,e^{-x}\,, & x \gg 1.\end{cases} \tag{B.5}$$

Further steps depend on the class of potentials, for which the consistency of the piecewise approximation should be carefully analyzed.

B.1. Small potential

The approximation (4.1) - (4.5) is efficient only if the ranges of validity of two asymptotics (respectively for small and big s) overlap with each other and the point s_* belongs to this overlap. Below we show that this requirement is satisfied for two classes of potentials satisfying the bounds (4.8) and (4.12).

The modified gradient expansion is applicable in this overlap range of s if

$$s\nabla\nabla V \ll V, \tag{B.6}$$

(cf. Eq.(2.17) with s replaced by effective cutoff $s = 1/V$). The applicability of the large s expansion in the same domain reads as

$$s\int dx\,V\Phi \gg \int dx\,\nabla_\mu\Phi\frac{1}{V-\Box}\nabla^\mu\Phi, \tag{B.7}$$

which means that the subleading term of the late time expansion (3.26) (quadratic in $\nabla_\mu\Phi$) is much smaller than the leading order term.

To implement these requirements we assume that $V(x)$ has a compact support of finite size R, (4.6), and its derivatives are bounded and satisfy the following estimate $\nabla\nabla V \sim V_0/R^2$, so that (B.6) reads as $sV_0/R^2 \ll V_0$, or

$$s \ll R^2. \tag{B.8}$$

To find out what does the criterion (B.7) mean let us make a further assumption, namely, that the potential V is small. In this case it can be disregarded in the Green's functions and $1/(\Box - V)$ can be replaced by $1/\Box$. Therefore the following estimates hold

$$\begin{aligned}
&\frac{1}{\Box - V} V(x) \sim \int_{|y|\leq R} dy\, \frac{1}{|x-y|^{d-2}} V(y) \sim \frac{1}{R^{d-2}} R^d V_0 \sim V_0 R^2,\\
&\int dx\, V\Phi \sim V_0 R^d,\\
&\int dx\, \nabla_\mu \Phi \frac{1}{V - \Box} \nabla^\mu \Phi \simeq V_0^2 R^{d+4}.
\end{aligned} \tag{B.9}$$

Roughly, every Green's function gives the factor R^2, every derivative – $1/R$, integration gives the volume of compact support R^d, etc. Applying these estimates to eq. (B.7) we get $sV_0R^d \gg V_0^2 R^{d+4}$, whence $s \gg V_0 R^4$. Combining this with (B.8) one gets the following range of overlap of our asymptotic expansions

$$R^2 \gg s \gg V_0 R^4, \tag{B.10}$$

whence it follows that this overlap domain is not empty only if

$$V_0 R^2 \ll 1. \tag{B.11}$$

Moreover, the assumption of disregarding the potential in the Green's function is also justified in this case since $V \sim V_0 \ll 1/R^2 \sim \Box$.

Now let us check whether s_* introduced above belongs to the overlap domain (B.10). Note that if it is really so, then $s_* V$ in Eq.(B.1) is much smaller than unity because in the overlap range one has $sV \sim sV_0 \ll R^2 V_0 \ll 1$. Hence the exponent in the left hand side of (B.1) can be expanded in powers of $s_* V$, and the resulting equation for s_* becomes[13]

$$\int dx\, \left(1 - s_* V + \frac{s_*^2}{2} V^2 + O\left((s_* V)^3\right)\right) = \int dx\, \left(1 - s_* V\Phi\right). \tag{B.12}$$

Its solution has the following form:

$$s_* \simeq 2\, \frac{\int dx\, V(1-\Phi)}{\int dx\, V^2} = 2\, \frac{\int dx\, V \frac{1}{V-\Box} V}{\int dx\, V^2}. \tag{B.13}$$

[13]Note that the quadratic term should be retained in the expansion of $e^{-s_* V}$ if we want to get a nontrivial solution for s_*.

Taking into account the estimates (B.9) we see that the point $s_* \sim R^2$ belongs to the upper edge of the interval (B.10). Late time expansions is fairly well satisfied here, but the small s expansion is on the verge of breakdown. At this level of generality it is hard to overstep the uncertainty of this estimate. There is a hope that numerical coefficients in more precise estimates (with concrete potentials) can be large enough to shift s_* to the interior of the interval (B.10) and, thus, make our approximation completely reliable.

Bearing in mind all these reservations let us proceed with the calculation of the effective action. We have to use the small $x = s_* V$ asymptotics (B.5) in the expression (B.4). This leaves us with the first two dominant terms in the right hand side of (B.4) plus the logarithmic term coming from the incomplete gamma function. Then we apply Eq.(B.13) to express the integral of $V(1-\Phi)$ in terms of s_* and that of V^2 and get

$$\varGamma_{>} \simeq \frac{1}{2\,(4\pi)^{d/2}} \int dx \left[-\frac{2}{d\,(d-2)} \frac{V^2}{s_*^{d/2-2}} + \frac{(-V)^{d/2}}{(d/2)!} \ln \frac{1}{s_* \mu^2} \right] - \varGamma_{\rm CW}, \qquad \text{(B.14)}$$

where the d-dimensional Coleman-Weinberg term $\varGamma_{\rm CW} = \varGamma_{\rm CW}(\mu^2)$, (2.15), was disentangled from the logarithm of the incomplete gamma function. Therefore, in the whole action $\bar{\varGamma} = \varGamma_{<} + \varGamma_{>}$ the Coleman-Weinberg term gets cancelled and the final answer reads as (4.10).

For generic potential satisfying the smallness bound (4.7) $s_* V \ll 1$, so that the first term in square brackets of (B.14) is much bigger than the logarithmic one, $V^2/s_*^{d/2-2} \gg V^{d/2}$ (confer Eq.(4.11)). However, in four dimensions, $d=4$, both terms become of the same order of magnitude and the first term takes the form of the finite counterterm $\sim \int d^4x\, V^2$ to ultraviolet divergences, which we disregard (or absorb in the redefinition of the μ^2-parameter). Therefore, in the four-dimensional case only the logarithmic term survives and gives rise to (4.8).

B.2. Big potential

Remarkably, the case of the small potentials (4.7) is not the only one when one can find a non-empty domain of overlap where both asymptotics for $\mathrm{Tr}\, K(s)$ are applicable. Namely, the opposite case of big potentials (in units of the inverse size of their support), (4.12), is equally good. The key observation here is that in this case the kernel of the Green's function $1/(\Box - V)$ can be replaced within the compact support of V by $-1/V$ ($\Box \sim 1/R^2 \ll V_0 \sim V$) and correspondingly

$$\frac{1}{\Box - V} V(x) \sim -\frac{1}{V} V = -1, \qquad \text{(B.15)}$$

$$\int dx\, \nabla_\mu \Phi \frac{1}{V - \Box} \nabla^\mu \Phi \simeq \frac{R^d}{V_0 R^2}. \qquad \text{(B.16)}$$

Therefore, the criterion of applicability of the late time expansion (B.7) becomes $s \gg 1/V_0^2 R^2$. Together with (B.8) it yields the new overlap range

$$R^2 \gg s \gg \frac{1}{V_0^2 R^2} \qquad \text{(B.17)}$$

which is obviously not empty if the potential satisfies (4.12).

To find s_* in this case we have to solve the equation (B.1) for the case when $s_* V$ is not anymore a small quantity. Since V is big, the exponent in (B.1) can be replaced by zero inside the compact support, $\exp(-s_* V(x)) \sim 0$, $|x| \leq R$, and by one outside of it where the potential vanishes, $\exp(-s_* V(x)) \sim 1$, $|x| > R$. Rewriting the integrals in both sides of the equation (B.1) as a sum of contributions of $|x| \leq R$ and $|x| > R$, we see that the contribution of the non-compact domain gets cancelled and the equation becomes

$$s_* \int\limits_{|x|\leq R} dx\, V\Phi \simeq \int\limits_{|x|\leq R} dx. \tag{B.18}$$

Then it follows that s_* is approximately given by the inverse of the function $V\Phi(x)$ *averaged* over the compact support of the potential

$$s_* \simeq \frac{1}{\big\langle V\Phi \big\rangle}, \tag{B.19}$$

where $\big\langle V\Phi \big\rangle$ is given by (4.14).

A qualitative estimate of $\big\langle V\Phi \big\rangle \sim V_0$ implies that $s_* \sim 1/V_0$ and it belongs to the middle of the interval (B.17). This makes the case of big potentials fairly consistent. On the other hand, the value of $\Phi(x)$ is close to zero inside the potential support (see (B.15)), so most likely the estimate for $\big\langle V\Phi \big\rangle$ is smaller by at least one power of the quantity $1/V_0 R^2$, which is the basic dimensionless small parameter in this case. Therefore the magnitude of s_* becomes bigger by one power of $V_0 R^2$, $s_* \simeq R^2$, which is again near the upper boundary of the overlap interval (B.17). Similarly to the small potential case, a more rigorous analysis is needed (maybe for more concretely specified potentials) to account for subtle edge effects at the boundary of compact support, which might shift the value of s_* to a safe region inside (B.17).

With the above estimate for $s_* \sim R^2$ the magnitude of $s_* V$ in the expression for the infrared part of the effective action (B.4) becomes big, $s_* V \sim s_* V_0 \sim V_0 R^2 \gg 1$. Therefore, we use the big x asymptotics (B.5) in (B.4) and get on account of (B.18) the contribution

$$\varGamma_> \simeq \frac{1}{(4\pi^2 s_*)^{d/2}}\, \frac{2 s_*}{d\,(d-2)} \int\limits_{|x|\leq R} dx\, V\Phi = \frac{1}{(4\pi)^{d/2}}\, \frac{2\,\big\langle V\Phi \big\rangle^{d/2}}{d\,(d-2)} \int\limits_{|x|\leq R} dx\,. \tag{B.20}$$

In this case the Coleman-Weinberg term is not cancelled in complete agreement with what we would expect for big potentials and the final result reads as (4.13).

C. Nonlocal form of the Gibbons-Hawking surface integral

To check the relation (5.16) we first calculate the metric variational derivative of its left hand side at finite $|x|$ and show that it is zero with the needed accuracy in

powers of the curvature. The variations of the linear, quadratic and cubic in curvature terms give respectively

$$\frac{\delta}{\delta g_{\mu\nu}}\int dx\, g^{1/2}\, R = -g^{1/2}\Big(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\Big), \tag{C.1}$$

$$\frac{\delta}{\delta g_{\mu\nu}}\int dx\, g^{1/2}\,\Big\{-R_{\mu\nu}\frac{1}{\Box}R^{\mu\nu}+\frac{1}{2}\,R\frac{1}{\Box}R\Big\}$$
$$= g^{1/2}\Big(R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}R\Big)+g^{1/2}\,J^{\mu\nu}+\mathrm{O}[\,R^3_{\mu\nu}\,], \tag{C.2}$$

$$\frac{\delta}{\delta g_{\mu\nu}}\int dx\, g^{1/2}\,\Big\{\frac{1}{2}\,R\Big(\frac{1}{\Box}R^{\mu\nu}\Big)\frac{1}{\Box}R_{\mu\nu}-R^{\mu\nu}\Big(\frac{1}{\Box}R_{\mu\nu}\Big)\frac{1}{\Box}R$$
$$+\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\Big(\nabla_\alpha\frac{1}{\Box}R\Big)\nabla_\beta\frac{1}{\Box}R$$
$$-2\,\Big(\nabla^\mu\frac{1}{\Box}R^{\nu\alpha}\Big)\Big(\nabla_\nu\frac{1}{\Box}R_{\mu\alpha}\Big)\frac{1}{\Box}R$$
$$-\,2\,\Big(\frac{1}{\Box}R^{\mu\nu}\Big)\Big(\nabla_\mu\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\nu\frac{1}{\Box}R_{\alpha\beta}\Big\} = -g^{1/2}\,J^{\mu\nu}+\mathrm{O}[\,R^3_{\mu\nu}\,], \tag{C.3}$$

where $J^{\mu\nu}=J^{\mu\nu}[\,g\,]$ is the following rather complicated nonlocal expression quadratic in Ricci tensor

$$J^{\mu\nu}= g^{\mu\nu}\,\Big\{-R^{\alpha\beta}\,\frac{1}{\Box}R_{\alpha\beta}+\Box\Big[\frac{1}{4}\Big(\frac{1}{\Box}R_{\alpha\beta}\Big)\frac{1}{\Box}R^{\alpha\beta}+\frac{1}{8}\Big(\frac{1}{\Box}R\Big)^2\Big]$$
$$-2\,\Big(\nabla^\gamma\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\frac{1}{\Box}R_{\beta\gamma}\Big\}$$
$$+\,g^{\mu\nu}\frac{1}{\Box}\,\Big\{\frac{1}{4}R^2-\frac{1}{2}R^2_{\alpha\beta}+\frac{1}{4}\Box R\,\frac{1}{\Box}R+\frac{1}{2}\Box R^{\alpha\beta}\,\frac{1}{\Box}R_{\alpha\beta}-\nabla_\alpha\nabla_\beta R\,\frac{1}{\Box}R^{\alpha\beta}$$
$$-R^{\alpha\beta}\,\nabla_\alpha\nabla_\beta\frac{1}{\Box}R+2\,\Big(\nabla^\alpha\nabla^\beta\frac{1}{\Box}R^{\gamma\delta}\Big)\nabla_\gamma\nabla_\delta\frac{1}{\Box}R_{\alpha\beta}\Big\}$$
$$-\frac{1}{2}\Big(\nabla^\mu\frac{1}{\Box}R\Big)\nabla^\nu\frac{1}{\Box}R-R^{\mu\nu}\,\frac{1}{\Box}R+\Big(\nabla^{(\mu}\frac{1}{\Box}R^{\nu)\alpha}\Big)\nabla_\alpha\frac{1}{\Box}R$$
$$-\Big(\nabla^{(\mu}\nabla_\alpha\frac{1}{\Box}R\Big)\frac{1}{\Box}R^{\nu)\alpha}-\Big(\nabla^\mu\frac{1}{\Box}R_{\alpha\beta}\Big)\nabla^\nu\frac{1}{\Box}R^{\alpha\beta}+2\,\Big(\frac{1}{\Box}R^{\alpha\beta}\Big)\nabla_\alpha\nabla_\beta\frac{1}{\Box}R^{\mu\nu}$$
$$+\nabla^{(\mu}\frac{1}{\Box}\Big\{R^{\nu)\alpha}\,\nabla_\alpha\frac{1}{\Box}R+\nabla_\alpha R\,\frac{1}{\Box}R^{\nu)\alpha}+2R_{\alpha\beta}\,\nabla^{\nu)}\frac{1}{\Box}R^{\alpha\beta}$$
$$+4\,\Big(\nabla^{\nu)}\nabla^\alpha\frac{1}{\Box}R^{\beta\gamma}\Big)\nabla_\gamma\frac{1}{\Box}R_{\alpha\beta}-4\,\Big(\nabla^\alpha\nabla^\beta\frac{1}{\Box}R^{\nu)\gamma}\Big)\nabla_\gamma\frac{1}{\Box}R_{\alpha\beta}\Big\}. \tag{C.4}$$

Here we systematically integrated by parts neglecting the surface terms which vanish for $\delta g_{\mu\nu}(x)$ with a compact support and took into account that the commutator of covariant derivatives contributes to the next order of the expansion in Ricci curvatures. This commutator contains the Riemannian tensor which in virtue of the Bianchi identities can be expressed as the nonlocal power series in Ricci tensor [11]

$$R^{\mu\nu\alpha\beta}=2\nabla^{[\mu}\nabla^\alpha\frac{1}{\Box}R^{\nu]\beta}-2\nabla^{[\mu}\nabla^\beta\frac{1}{\Box}R^{\nu]\alpha}+\mathrm{O}[\,R^2_{\mu\nu}\,]. \tag{C.5}$$

As a result the sum of three terms (C.1)-(C.3) cancels in the quadratic approximation (corresponding to the cubic approximation for $\Sigma[\,g\,]$), and the expression (5.16) turns out to be topologically invariant with respect to local metric variations.

To obtain the dependence of $\Sigma = \Sigma[\,g_\infty\,]$ on the asymptotic behavior of metric at infinity one should, instead of the functional derivative, calculate its variation in the class of $\delta g_{\mu\nu}(x) \sim 1/|x|^{d-2}$. Repeating the same steps as above, one finds that only surface terms at infinity will survive and, moreover, all surface integrals generated by nonlinear terms of (5.16) vanish in view of this falloff. The only nonvanishing surface integral comes from the Einstein part linear in the curvature and reads

$$\delta_g\Sigma = \int dx\, g^{1/2}\, \big(\nabla^\mu\nabla^\nu\delta g_{\mu\nu} - g^{\alpha\beta}\Box\delta g_{\alpha\beta}\big) = \int_\infty d\sigma^\mu\; \delta^{\alpha\beta}(\partial_\alpha\delta g_{\beta\mu} - \partial_\mu\delta g_{\alpha\beta}), \quad \text{(C.6)}$$

where the surface integration measure and the metric contracting indices can be taken flat-space ones again in view of falloff properties of $g_{\mu\nu}(x)$ and $\delta g_{\mu\nu}(x)$ at $|x| \to \infty$. Thus, the right hand side can be easily functionally integrated to give (5.16)[14].

D. Metric dependence in the subleading order

Using the expression (5.2) for $K(s\,|\,x,y)$ and the relation $f^{\mu\nu}(\nabla_x,\nabla_y)\,\sigma(x,y)\,\big|_{y=x} = g^{\mu\nu}$ one has

$$\begin{aligned}\frac{\delta \mathrm{Tr}\,K(s)}{\delta g_{\mu\nu}(x)} &= -s\, g^{1/2}(x)\, f^{\mu\nu}(\nabla_x,\nabla_y) K(s|x,y)\,\Big|_{x=y}\\ &= \frac{g^{1/2}(x)}{(4\pi s)^{d/2}}\,\Big[\,\frac{1}{2}\, g^{\mu\nu}\,\Omega(s|\,x,x) - s\, f^{\mu\nu}(\nabla_x,\nabla_y)\,\Omega(s|\,x,y)\,\Big|_{y=x}\Big], \qquad \text{(D.1)}\end{aligned}$$

so that the metric variational derivative for W_1 looks more complicated than (5.12) and has a contribution from the leading order

$$\frac{\delta W_1}{\delta g_{\mu\nu}} = \frac{1}{2}\, g^{1/2} g^{\mu\nu}\Phi^2(x) - g^{1/2} f^{\mu\nu}(\nabla_x,\nabla_y)\,\Omega_1(x,y)\,\Big|_{y=x}. \qquad \text{(D.2)}$$

Checking this relation for W_1, given by Eq. (5.39) with metric-independent $W_1^{\rm metric}$, begins with the calculation of $\Omega_1(x,y)$-contribution to the right hand side of (D.2)

$$\begin{aligned}-g^{1/2} f^{\mu\nu}(\nabla_x,\nabla_y)\,\Omega_1(x,y)\,\Big|_{y=x} &= -\frac{1}{2}\, g^{1/2} f^{\mu\nu}(\nabla_x,\nabla_y)\,\Big[\,\psi(x,y)+\psi(y,x)\,\Big]_{y=x}\\ &+\frac{1}{2}\, g^{1/2} f^{\mu\nu}(\nabla_x,\nabla_y)\,\left[\,\frac{1}{F_x}\,\overrightarrow{F}_x\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,\overleftarrow{F}_y\,\overleftarrow{\frac{1}{F_y}}\,\right]_{y=x}, \qquad \text{(D.3)}\end{aligned}$$

[14]Alternatively, Eq.(5.16) can be derived by expanding its left hand side in powers of metric perturbations, $g_{\mu\nu}(x) = \delta_{\mu\nu} + h_{\mu\nu}(x)$, and directly observing the cancellation of the bulk part, while the nonvanishing surface terms reduce to the noncovariant form of the Gibbons-Hawking surface integral.

where $\psi(x,y)$ is given by Eq.(5.33) and we use obvious abbreviations $F_x = F(\nabla_x)$, etc. From the structure of the operator $f^{\mu\nu}(\nabla_x,\nabla_y)$, (5.11), it follows that

$$f^{\mu\nu}(\nabla_x,\nabla_y)\Big[\psi(x,y)-\psi(y,x)\Big]_{y=x}$$
$$=\frac{1}{2}\,g^{\mu\nu}\Big[F_x\,\psi(x,y)-F_x\,\Psi(y,x)\Big]_{y=x}, \qquad \text{(D.4)}$$

and the two terms here can be simplified to

$$F_x\,\psi(x,y)\Big|_{y=x}=\Phi(x)\,\Big[\overrightarrow{F}_x\;\Phi(x)\,\sigma(x,y)\Big]_{y=x}-d\,\Phi^2(x)=0, \qquad \text{(D.5)}$$

$$F_x\,\psi(y,x)\Big|_{y=x}=\frac{1}{F_x}\,\overrightarrow{F}_x\,[\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,]\,\overleftarrow{F}_y\Big|_{y=x}. \qquad \text{(D.6)}$$

Therefore, the expression (D.2) takes the form in which $\psi(y,x)$ enters without symmetrization of its arguments

$$-g^{1/2}\,f^{\mu\nu}(\nabla_x,\nabla_y)\,\Omega_1(x,y)\Big|_{y=x}=\frac{1}{4}\,g^{1/2}g^{\mu\nu}\,\frac{1}{F_x}\,\overrightarrow{F}_x\,[\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,]\,\overleftarrow{F}_y\Big|_{y=x}$$
$$-\,g^{1/2}f^{\mu\nu}(\nabla_x,\nabla_y)\;\psi(y,x)\Big|_{y=x}$$
$$+\frac{1}{2}\,g^{1/2}f^{\mu\nu}(\nabla_x,\nabla_y)\,\left[\frac{1}{F_x}\,\overrightarrow{F}_x\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,\overleftarrow{F}_y\,\overleftarrow{\frac{1}{F_y}}\right]_{y=x}. \qquad \text{(D.7)}$$

On the other hand, the metric variational derivative of the nonlocal functional W_1, (5.39), with the metric independent $W_1^{\rm metric}$, (5.43), consists of the following five terms

$$\frac{1}{2}\,\frac{\delta}{\delta g_{\mu\nu}(x)}\int dz\,g^{1/2}(z)\,\frac{1}{F_z}\,\overrightarrow{F}_z\;\Phi(z)\,\sigma(z,y)\,\Phi(y)\,\overleftarrow{F}_y\;\Big|_{y=z}$$
$$=\frac{1}{2}\int dz\,g^{1/2}(z)\,\frac{1}{F_z}\,\overrightarrow{F}_z\;\Phi(z)\,\frac{\delta\sigma(z,y)}{\delta g_{\mu\nu}(x)}\,\Phi(y)\,\overleftarrow{F}_y\;\Big|_{y=z}$$
$$+\frac{1}{4}\,g^{1/2}g^{\mu\nu}\,\frac{1}{F_x}\,\overrightarrow{F}_x\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,\overleftarrow{F}_y\;\Big|_{y=x}$$
$$+\frac{1}{2}\,g^{1/2}f^{\mu\nu}(\nabla_x,\nabla_y)\,\Big(\frac{1}{F_x}\,\overrightarrow{F}_x\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,\overleftarrow{F}_y\,\overleftarrow{\frac{1}{F_y}}\,\Big)_{y=x}$$
$$-\,g^{1/2}f^{\mu\nu}(\nabla_x,\nabla_y)\,\Big(\,\Phi(x)\,\sigma(x,y)\,\Phi(y)\,]\,\overleftarrow{F}_y\,\overleftarrow{\frac{1}{F_y}}\,\Big)_{y=x}$$
$$+\int dz\,g^{1/2}(z)\,\frac{1}{F_z}\,\overrightarrow{F}_z\;G(z,x)\,f^{\mu\nu}(\overrightarrow{\nabla}_x,\overleftarrow{\nabla}_x)\,\Phi(x)\,\sigma(z,y)\,\Phi(y)\,\overleftarrow{F}_y\Big|_{y=z}. \qquad \text{(D.8)}$$

The first three terms correspond to the variation of the world function, measure $g^{1/2}(z)$ and Green's function $1/F_z$ respectively, whereas the last two ones – to the variation of operators $(F_z,\,F_y)$ and the functions $(\Phi(z),\Phi(y))$.

In the first term one can integrate by parts so that the operator F_z will act to the left and "cancel" the Green's function. The corresponding surface term will be zero

for the following reason. The variational derivative $\delta\sigma(z,y)/\delta g_{\mu\nu}(x)$ is not vanishing at infinity, but on the sphere of infinitely growing radius $|z|\to\infty$ it has a support only at the point where the geodesic emanating from the point y and passing through x punctures this sphere. The contribution of this point is suppressed to zero by the angular measure $\sim\delta(\theta)\sin^{d-1}\theta$, where θ is the longitudinal spherical angle (with the origin at the north pole $\theta=0$ coinciding with this point). Integration by parts then yields the local expression

$$\frac{1}{2}\int dz\, g^{1/2}(z)\,\frac{1}{F_z}\,\overrightarrow{F}_z\;\Phi(z)\,\frac{\delta\sigma(z,y)}{\delta g_{\mu\nu}(x)}\,\Phi(y)\,\overleftarrow{F}_y\Big|_{y=z}$$
$$=\frac{1}{2}\int dz\, g^{1/2}(z)\,\Phi(z)\,F(\nabla_y)\,\frac{\delta\sigma(z,y)}{\delta g_{\mu\nu}(x)}\,\Phi(y)\,\Big|_{y=z}=\frac{1}{2}\,g^{1/2}g^{\mu\nu}\,\Phi^2(x),\quad \text{(D.9)}$$

based on simple coincidence limits

$$\nabla^y_\alpha\,\frac{\delta\sigma(y,z)}{\delta g_{\mu\nu}(x)}\,\Big|_{y=z}=0,$$
$$\Box_y\,\frac{\delta\sigma(y,z)}{\delta g_{\mu\nu}(x)}\,\Big|_{y=z}=\frac{\delta}{\delta g_{\mu\nu}}\Big(\Box_y\sigma(y,z)\,|_{z=y}\Big)-\frac{\delta\,\Box_y}{\delta g_{\mu\nu}}\,\sigma(y,z)\,\Big|_{y=z}=g^{\mu\nu}\delta(z,x).\quad \text{(D.10)}$$

This expression obviously reproduces the first term on the right hand side of Eq.(D.2).

In the last term of (D.8) integration by parts over z is again possible without extra surface terms because $G(z,x)\sigma(z,y)\,\Phi(y)\,\overleftarrow{F}_y\sim 1/|z|^{d-3}$, therefore the (z,y)-part of the integrand reduces to the local coincidence limit $\sigma(z,y)\,\Phi(y)\,\overleftarrow{F}_y\,\big|_{y=z}=d\,\Phi(z)$ and the result looks

$$\int dz\, g^{1/2}(z)\,\frac{1}{F_z}\,\overrightarrow{F}_z\;G(z,x)\,f^{\mu\nu}(\overrightarrow{\nabla}_x,\overleftarrow{\nabla}_x)\,\Phi(x)\,\sigma(z,y)\,\Phi(y)\,\overleftarrow{F}_y\,\Big|_{y=z}$$
$$=g^{1/2}f^{\mu\nu}(\nabla_x,\nabla_y)\,\left[d\,\Phi(x)\,\Phi(y)\overleftarrow{\frac{1}{F_y}}\right]_{y=x}.\quad \text{(D.11)}$$

The fourth term of (D.8) together with (D.11) equals $-f^{\mu\nu}(\nabla_x,\nabla_y)\,\psi(y,x)\,\big|_{y=x}$. Thus the last four terms of (D.8) reproduce the contribution of $\Omega_1(x,y)$, Eq.(D.7), to the metric variational derivative (D.2). Taken together with (D.9) this completes the proof of the equation (D.2) for the subleading coefficient W_1 given by (5.39) with a constant (metric-independent) $W_1^{\rm metric}$.

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